

ENVELOPES AND IMPRINTS IN CATEGORIES

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Introduction

In 1972 J. L. Taylor introduced in his paper [36] an operation, which associates to an arbitrary topological algebra A a new topological algebra¹ A^\heartsuit called by A. Ya. Helemskii later “the Arens-Michael envelope of A ” [14]. Immediately after that in his next paper [37] Taylor gave an amusing formula² which suggests an unexpectedly simple way to formalize the heuristically evident connection between algebraic geometry and complex analysis:

$$\mathcal{P}(\mathbb{C}^n)^\heartsuit = \mathcal{O}(\mathbb{C}^n) \quad (\text{A})$$

(here $\mathcal{P}(\mathbb{C}^n)$ and $\mathcal{O}(\mathbb{C}^n)$ are the algebras of polynomials and, respectively, of holomorphic functions on the complex space \mathbb{C}^n). Despite this promising application, up to the end of the century Taylor’s construction did not manifest itself in mathematical literature, and only recently the interest to the operation $A \mapsto A^\heartsuit$ appeared again in A.Yu.Pirkovskii’s papers on “holomorphic non-commutative geometry” [26], [27]. In particular, in [27] formula (A) was generalized to the case of arbitrary affine algebraic manifold M :

$$\mathcal{P}(M)^\heartsuit = \mathcal{O}(M). \quad (\text{B})$$

This identity very soon was applied by the author in [3] to the construction of a generalization of Pontryagin’s duality from the category of commutative compactly generated Stein groups to the category of arbitrary compactly generated Stein groups with the algebraic connected component of identity. The idea of the duality suggested in [3] is illustrated by the diagram

$$\begin{array}{ccc} \mathcal{O}^*(G) & \xrightarrow{\heartsuit} & \mathcal{O}_{\text{exp}}^*(G) \\ \uparrow \star & & \downarrow \star \\ \mathcal{O}(G) & \xleftarrow{\heartsuit} & \mathcal{O}_{\text{exp}}(G) \end{array} \quad (\text{C})$$

where G is a group of the described class, $\mathcal{O}(G)$ the algebra of holomorphic functions on G , $\mathcal{O}_{\text{exp}}(G)$ its subalgebra, consisting of functions of exponential type, $A \mapsto A^\heartsuit$ the operation of taking Arens-Michael envelope, and $X \mapsto X^*$ the operation of passage to the dual stereotype space in the sense of [2], i.e. to the space of linear continuous functionals with the topology of uniform convergence on totally bounded sets (in this case this is equivalent to the uniform convergence on compact sets).

One can call duality, presented in diagram (C), the *complex geometry duality*, having in mind the class of objects under consideration. The obtained theory for the described class of groups contrasts with the other existing theories in the following two points. First, its enveloping category (in which the group algebras lie) consists of Hopf algebras. And, second, the diagram (C) suggests a natural way for constructing the analogous dualities for the “other geometries”, in particular, for differential geometry and for topology: one should just replace the Arens-Michael envelope in diagrams analogous to (C) with some other envelopes (and this automatically leads to the replacing of the constructions in the corners of the diagram with some proper analogs from analysis). This alleged connection between different dualities in geometry and different envelopes of topological algebras was recently vouched by another example: in the work by J. N. Kuznetsova [21] the Arens-Michael envelope was replaced by the C^* -envelope³, and this immediately led to a variant of *topological duality*, where the Stein groups are replaced by the Moore groups, and the algebras $\mathcal{O}(G)$ and $\mathcal{O}_{\text{exp}}(G)$, respectively, by the algebra $\mathcal{C}(G)$ of continuous functions on G and the algebra $\mathcal{K}(G)$ of coefficients of norm-continuous representations of G .

It is interesting (and predictable), that in these theories the classical Fourier and Gelfand transforms are interpreted as envelopes with respect to the prescribed class of algebras (see below Theorems 4.22, 4.29 and 4.28).

This paper is intended as a part of the program, suggested in [3]. We discuss here the question (which remained open up to the last time), how one should define envelopes in general category theory, and under which conditions they exist and are functors? We suggest a natural definition (from our point of view) and establish some wide necessary and sufficient conditions for existence of envelopes and their dual constructions, which we call imprints. As applications, we show that in the categories **Ste** of stereotype spaces, and **Ste**[®] of stereotype algebras the envelopes and the imprints exist in a very wide class of situations. We plan to use these results further in building the above mentioned duality theory for differential geometry.

¹We use the notation A^\heartsuit from [3].

²Taylor mentions this fact in passing on pages 207 and 251 in [37].

³We give the definition of C^* -envelopes on page 117.

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Agreements and Notation

Everywhere in category theory we use the terminology of textbooks [8], [38] and of handbook [11], and as a set-theoretic fundament for the notion of category we choose the Morse-Kelly theory [16]. We say that a category K is

- *injectively (projectively) complete*, if each functor $K : M \rightarrow K$ from a small category M (i.e. a category where the class of morphisms is a set) has an injective (projective) limit,
- *complete*, if it is injectively and projectively complete,
- *finitely injectively (projectively) complete*, if each functor $K : M \rightarrow K$ from a finite category M (i.e. a category where the class of morphisms is a finite set) has an injective (projective) limit,
- *finitely complete*, if it is finitely injectively complete and finitely projectively complete,
- *linearly complete*, if any functor from a linearly ordered set to K has injective and projective limits.

The choice of the Morse-Kelly theory as a fundament justifies the following definition: we say that a category K is

- *injectively determined*, if there is a map, which to any functor $F : M \rightarrow K$ from any small category M assigns its injective limit,
- *projectively determined*, if there exists a map, which to any functor $F : M \rightarrow K$ from any small category M assigns its projective limit.
- *determined*, if it is both injectively and projectively determined.

Certainly, if a category is injectively (projectively) determined, it is injectively (projectively) complete. But inside the Morse-Kelly (or von Neumann-Bernays-Gödel) set theory it is apparently impossible to prove the inverse, due to the absence of the analog of the choice axiom for “families of classes”.

In the theory of topological vector spaces we follow textbooks [10] and [34], and in the theory of stereotype spaces and algebras author's papers [2] and [3]. By *topological algebra* we mean locally convex topological algebra in the spirit of textbook [22], i.e. locally convex space A over the field \mathbb{C} , endowed with associative multiplication which is *separately continuous* and has unit.

We use also the following notations. First, for any morphism $\varphi : X \rightarrow Y$ in an arbitrary category the symbols $\text{Dom}(\varphi)$ and $\text{Ran}(\varphi)$ mean respectively the domain and the range of φ , i.e. $\text{Dom}(\varphi) = X$ and $\text{Ran}(\varphi) = Y$. Second, for any locally convex space X the symbol $\mathcal{U}(X)$ denotes the system of all neighborhoods of zero in X . And, third, if a topological space Y is imbedded into a topological space X (injectively, but not necessarily in such a way that the topology of Y is inherited from X), and A is a subset in Y , then to distinguish the closure of A in Y from its closure in X , we denote the first one by \overline{A}^Y , and the second by \overline{A}^X .

Besides this we say that a subset M in a locally convex space X is *total (in X)*, if its linear span $\text{Span } M$ is dense in X :

$$\overline{\text{Span } M}^X = X.$$

§ 0 Some preliminary facts from the category theory

(a) Skeletally small graphs

Graphs. Recall that an *oriented graph* is a set V with a given subset Γ in its cartesian square $V \times V$. The elements of V are called vertices, and the elements of Γ edges of this graph. An oriented graph is said to be *reflexive*, if for each vertex $x \in V$ the edge (x, x) belongs to Γ , and *transitive*, if for any two edges (x, y) and (y, z) from Γ the pair (x, z) also belongs to Γ . Obviously, every reflexive transitive oriented graph is a (small) category, where objects and morphisms are respectively the vertices and the edges (the multiplication of edges (x, y) and (y, z) is the edge (x, z) , and local identities 1_x are (x, x)). The characteristic property of such categories (apart from the requirement if being small), is that the sets of morphisms $\text{Mor}(A, B)$ always contain at most one element. This justifies the following definition.

- A *graph* is a category \mathbf{K} (not necessarily small), where each set of morphisms $\text{Mor}(A, B)$ contains at most one element:

$$\forall A, B \in \text{Ob}(\mathbf{K}) \quad \text{card Mor}(A, B) \leq 1. \quad (0.1)$$

Clearly, this condition is equivalent to establishing the structure of (reflexive and transitive) oriented graph at the class $\text{Ob } \mathbf{K}$ of objects of the category \mathbf{K} (with the difference that $\text{Ob } \mathbf{K}$ is not necessarily a set, but just a class).

Properties of graphs:

- 1°. In any graph a morphism $\varphi : A \rightarrow B$ is an isomorphism, iff there exists an arbitrary morphism to the reverse direction $\psi : A \leftarrow B$,

$$\forall \varphi \in \text{Mor}(A, B) \quad \left(\varphi \in \text{Iso} \iff \exists \psi \in \text{Mor}(B, A) \right). \quad (0.2)$$

- 2°. In any graph a composition of morphisms is an identity iff the same remains true after the replacing the factors:

$$\psi \circ \varphi = 1 \iff \varphi \circ \psi = 1. \quad (0.3)$$

- 3°. In any graph a composition of morphisms $\psi \circ \varphi$ is an isomorphism iff both ψ and φ are isomorphisms:

$$\psi \circ \varphi \in \text{Iso} \iff \psi \in \text{Iso} \ \& \ \varphi \in \text{Iso}. \quad (0.4)$$

Proof. 1. If $\varphi : A \rightarrow B$ and $\psi : A \leftarrow B$, then $\psi \circ \varphi$ acts from A into A , so it must coincide with 1_A . Similarly, $\varphi \circ \psi$ acts from B into B , so it must coincide with 1_B .

2. From $\psi \circ \varphi = 1$ it follows that $\text{Ran } \varphi = \text{Dom } \psi$ and $\text{Ran } \psi = \text{Dom } \varphi$, and after that we apply the same reasoning as in step 1.

3. If $\omega = \psi \circ \varphi \in \text{Iso}$, then $\psi \circ \varphi \circ \omega^{-1} = 1$, so by (0.3), $\varphi \circ \omega^{-1} \circ \psi = 1$, hence, $\psi \in \text{Iso}$, and finally $\varphi = \psi^{-1} \circ \omega \in \text{Iso}$. \square

Partially ordered classes. Every partially ordered set I can be considered as a category, where objects are elements of this set, and morphisms are pairs (i, j) , for which $i \leq j$. Such categories \mathbf{K} , of course, are special cases of graphs, since every set of morphisms $\text{Mor}(A, B)$ here contains at most one element (i.e. (0.1) holds). But in addition (and this property distinguishes the partially ordered sets among all graphs), for $A \neq B$ the existence of a morphism $\varphi : A \rightarrow B$ automatically make impossible the existence of any morphisms $\psi : A \leftarrow B$. This justifies the following definition.

- A *partially ordered class* is a graph, where the existence of opposite morphisms $\varphi : A \rightarrow B$ and $\psi : A \leftarrow B$ is possible only if $A = B$ (and then $\varphi = \psi = 1_A$). In other words,

$$\forall A \neq B \in \text{Ob}(\mathbf{K}) \quad \text{Mor}(A, B) \neq \emptyset \implies \text{Mor}(B, A) = \emptyset. \quad (0.5)$$

Obviously, these requirements are equivalent to the establishing the structure of partial order at the class $\text{Ob } \mathbf{K}$ of objects of the category \mathbf{K} (again. like in the previous definition, with the difference that $\text{Ob } \mathbf{K}$ is not necessarily a set, but just a class).

Example 0.1. *Category of ordinal numbers* \mathbf{Ord} . The class \mathbf{Ord} of all ordinal numbers with its natural order (see e.g. [16]) is an example of a partially ordered class which is not a set.

Proposition 0.1. *In a partially ordered set only local identities are isomorphisms:*

$$\forall \varphi \in \text{Mor}(A, B) \quad \left(\varphi \in \text{Iso} \iff A = B \ \& \ \varphi = 1_A \right).$$

Proof. The identity $A = B$ follows from the fact that $\text{Mor}(A, B) \neq \emptyset$ and $\text{Mor}(B, A) \neq \emptyset$, and the identity $\varphi = 1_A$ from the fact that φ and 1_A are colinear arrows in a graph. \square

Skeleton. A class S of objects of a category K is called a *skeleton* of K , if every object in K is isomorphic to an exactly one object of S . In other words, S satisfies the following two requirements:

- 1) elements of S are isomorphic only if they coincide:

$$\forall X, Y \in S \quad (X \cong Y \Leftrightarrow X = Y);$$

- 2) there exists a map $G : \text{Ob}(K) \rightarrow S$ from the class of objects of K to the class S such that

$$\forall X \in \text{Ob}(K) \quad X \cong G(X).$$

The skeleton S is usually endowed with the structure of a full subcategory in K . Then any two skeletons in K (if exist) are isomorphic (as categories), any category K is equivalent to its skeleton S (if S exists), and any two categories K and L with skeletons are equivalent, iff their skeletons are isomorphic (as categories).

- A category K is said to be
 - *skeletal*, if any two isomorphic objects coincide there (this is equivalent to the requirement that K is a skeleton for itself),
 - *skeletally small*, if it has a skeleton, which is a set.

Example 0.2. Each partially ordered class is a skeletal category (since as we already noticed only local identities are isomorphisms there), but not vice versa. For instance, the category of all finite sets of the form $\{0, \dots, n\}$, $n \in \mathbb{Z}_+$, (with arbitrary maps as morphisms) is skeletal, but it is not a partially ordered class, since a set $\{0, \dots, n\}$ can have many bijections onto itself.

Transfinite chain condition.

- Let us say that a (covariant or contravariant) functor $F : \mathbf{Ord} \rightarrow K$ is *stabilized*, if it satisfies the following two equivalent conditions:
 - (i) there exists an ordinal number $k \in \mathbf{Ord}$ such that

$$\forall l \geq k \quad F(k, l) \in \text{Iso}$$

- (ii) there exists an ordinal number $k \in \mathbf{Ord}$ such that

$$\forall l, m \quad \left(k \leq l \leq m \implies F(l, m) \in \text{Iso} \right)$$

Proof of equivalence. The implication (i) \Leftarrow (ii) is obvious, so we need to prove only (i) \Rightarrow (ii). Let F be a covariant functor (the case of a contravariant functor is considered similarly). If (i) holds, then for $k \leq l \leq m$ we have:

$$\underbrace{F(k, m)}_{\text{Iso}} = F(l, m) \circ \underbrace{F(k, l)}_{\text{Iso}} \implies \underbrace{F(k, m)}_{\text{Iso}} \circ \underbrace{F(k, l)^{-1}}_{\text{Iso}} = F(l, m) \implies F(l, m) \in \text{Iso}$$

□

Remark 0.1. If a category K is a partially ordered class, then by Proposition 0.1, for a functor $F : \mathbf{Ord} \rightarrow K$ the isomorphisms in (i) and (ii) become local identities:

- (i)' there exists an ordinal number $k \in \mathbf{Ord}$ such that

$$\forall l \geq k \quad F(k, l) = 1_{F(l)}$$

- (ii)' there exists an ordinal number $k \in \mathbf{Ord}$ such that

$$\forall l, m \quad \left(k \leq l \leq m \implies F(l, m) = 1_{F(l)} \right)$$

Theorem 0.1 (transfinite chain condition). *Every functor $F : \mathbf{Ord} \rightarrow K$ into an arbitrary skeletally small graph K is stabilized.*

We will need the following

Lemma 0.1. *In the class \mathbf{Ord} of ordinal numbers there is no a cofinal subclass, which is a set.*

Proof. If K is a cofinal subclass in \mathbf{Ord} , then \mathbf{Ord} becomes a union of a family of sets, indexed by elements of K :

$$\mathbf{Ord} = \bigcup_{k \in K} \{i \in \mathbf{Ord} : i \leq k\}.$$

Hence if K is a set, then \mathbf{Ord} must also be a set, but this is not true. \square

Corollary 0.1. *For any directed set I each monotone map $F : I \rightarrow \mathbf{Ord}$ has a least upper bound in \mathbf{Ord} .*

Proof. It is sufficient to note here that the image $F(I)$ is bounded in \mathbf{Ord} . And this in its turn follows from the fact that $F(I)$ is a set, and thus cannot be a cofinal subclass in \mathbf{Ord} . \square

Proof of Theorem 0.1. Let $F : \mathbf{Ord} \rightarrow \mathbf{K}$ be a (covariant or contravariant) functor into a skeletally small graph \mathbf{K} . Suppose that it is not stabilized, i.e. for any ordinal number $i \in \mathbf{Ord}$ there is an ordinal number $j \in \mathbf{Ord}$ such that $F(i, j) \notin \mathbf{Iso}$. Let us construct a transfinite sequence of ordinal numbers $\{k_i; i \in \mathbf{Ord}\} \subseteq \mathbf{Ord}$ according to the following rules:

0) We set $k_0 = 0$.

1) If for some ordinal number $j \in \mathbf{Ord}$ all the ordinal numbers k_i with the smaller indices $\{k_i; i < j\}$ are already chosen, then we consider two cases:

— if j is an isolated ordinal, i.e. $j = i + 1$ for some $i < j$, then we take k_j with the properties

$$k_i < k_{i+1} = k_j, \quad F(k_i, k_{i+1}) = F(k_i, k_j) \notin \mathbf{Iso}$$

(k_j exists due to our assumption that F is not stabilized),

— if j is a limit ordinal, i.e. $j \neq i + 1$ for any $i < j$, then we take k_j as the least upper bound of k_i :

$$k_j = \lim_{i \rightarrow j} k_i = \sup_{i < j} k_i$$

(it exists due to Corollary 0.1).

We obtain a transfinite sequence $i \in \mathbf{Ord} \mapsto k_i \in \mathbf{Ord}$ with the following properties:

(i) It is cofinal in \mathbf{Ord} , since $i \leq k_i$ for any $i \in \mathbf{Ord}$.

(ii) For $i < j$ we have $F(k_i, k_j) \notin \mathbf{Iso}$, since

$$i < j \implies i + 1 \leq j \implies F(k_i, k_j) = F(k_{i+1}, k_j) \circ \underbrace{F(k_i, k_{i+1})}_{\substack{\cong \\ \mathbf{Iso}}} \xrightarrow{(0.4)} F(k_i, k_j) \notin \mathbf{Iso}$$

(we suppose here that F is a covariant functor, but for a contravariant one the reasoning is the same).

Now let $S \subseteq \mathbf{K}$ be a skeleton of \mathbf{K} . For any $i \in \mathbf{Ord}$ we consider the object $G(i) \in S$ such that

$$G(i) \cong F(k_i).$$

Suppose now that $G(i) = G(j)$ for some $i \leq j$. Then the morphism $F(k_i, k_j) : G(i) \rightarrow G(j)$ must coincide with the local identity $1_{G(i)} = 1_{G(j)}$, since the category \mathbf{S} is a graph, and therefore it cannot have two different colinear morphisms. Thus, $F(k_i, k_j)$ must be an isomorphism, and, by (ii), this is possible only if $i = j$. So we obtain that the map $G : \mathbf{Ord} \rightarrow S$ is injective. On the other hand, it turns the class \mathbf{Ord} into the set S , and this is impossible. \square

(b) Monomorphisms and epimorphisms

The widely used in the category theory notions of monomorphism and epimorphism have several variations, and two of them, the so-called immediate and strong mono- and epimorphisms, will be important for us further. As the reader will see, we will accentuate the analogy between mono/epimorphisms from the one hand and strong mono/epimorphisms from the other. In the cases, where due to this analogy the proofs becomes identical (up to the substitution of the epithet “strong” into the proper places, like in the results about categories $\mathbf{SMono}(X)$ and $\mathbf{SEpi}(X)$), as well as in the elementary propositions we omit the proofs.

We start with monomorphisms and epimorphisms. Recall that a morphism $\varphi : X \rightarrow Y$ is called

- a *monomorphism*, if any equality $\varphi \circ \alpha = \varphi \circ \beta$ implies $\alpha = \beta$;
- an *epimorphism*, if any equality $\alpha \circ \varphi = \beta \circ \varphi$ implies $\alpha = \beta$;
- a *bimorphism*, if it is a monomorphism and an epimorphism.

Example 0.3. In any graph K every morphism is a bimorphism. Indeed, if $\varphi \circ \alpha = \varphi \circ \beta$, then, since α and β are colinear, they coincide, $\alpha = \beta$. So φ is a monomorphism. Similarly, it is an epimorphism.

Proposition 0.2. A composition of two monomorphisms (respectively, two epimorphisms) is a monomorphism (respectively, an epimorphism).

Properties of mono- and epimorphisms:

- 1°. If $\varphi \circ \mu$ is a monomorphism, then μ is a monomorphism as well.
- 2°. If $\mu \circ \varphi$ is an isomorphism, and μ a monomorphism, then μ and φ are isomorphisms.
- 3°. If $\varepsilon \circ \varphi$ is an epimorphism, then ε is an epimorphism as well.
- 4°. If $\varphi \circ \varepsilon$ is an isomorphism, and ε an epimorphism, then φ and ε are isomorphisms.

The category of monomorphisms $\mathbf{Mono}(X)$ and systems of subobjects. Let X be an object in a category K . We denote by $\mathbf{Mono}(X)$ the class of all monomorphisms with X as a range. It is a category, where a morphism $\rho \xrightarrow{\varkappa} \sigma$ from an object $\rho \in \mathbf{Mono}(X)$ into an object $\sigma \in \mathbf{Mono}(X)$, i.e. a monomorphism $\rho : A \rightarrow X$ into a monomorphism $\sigma : B \rightarrow X$, is an arbitrary morphism $\varkappa : A \rightarrow B$ in K such that the following diagram is commutative:

$$\begin{array}{ccc} A & & \\ \downarrow \varkappa & \searrow \rho & \\ B & & X \\ & \nearrow \sigma & \end{array} \quad (0.6)$$

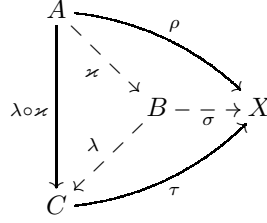
Actually, this diagram in the initial category K can be considered as a morphism $\rho \xrightarrow{\varkappa} \sigma$ in the category $\mathbf{Mono}(X)$. A composition of such morphisms $\rho \xrightarrow{\varkappa} \sigma$ and $\sigma \xrightarrow{\lambda} \tau$, i.e. of diagrams

$$\begin{array}{ccc} A & & \\ \downarrow \varkappa & \searrow \rho & \\ B & & X \\ & \nearrow \sigma & \end{array} \quad \begin{array}{ccc} B & & \\ \downarrow \lambda & \searrow \sigma & \\ C & & X \\ & \nearrow \tau & \end{array}$$

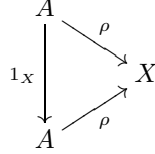
is a morphism $\rho \xrightarrow{\lambda \circ \varkappa} \tau$, i.e. a diagram

$$\begin{array}{ccc} A & & \\ \downarrow \lambda \circ \varkappa & \searrow \rho & \\ C & & X \\ & \nearrow \tau & \end{array}$$

One can conceive it as a result of splicing of the initial diagrams along the common edge σ , adding the arrow of composition $\varkappa \circ \lambda$, and then throwing away the vertex B together with all its incidental edges:



Of course, local identities in $\text{Mono}(X)$ are diagrams of the form



Remark 0.2. The composition of morphisms in $\text{Mono}(X)$ can be defined in two ways. In our definition this operation is connected with the composition in \mathbf{K} through the following identity:

$$\lambda \circ_{\text{Mono}(X)} \varkappa = \lambda \circ_{\mathbf{K}} \varkappa.$$

Theorem 0.2. For any object X the category $\text{Mono}(X)$ is a graph.

Proof. We should verify that for any two objects $\rho : A \rightarrow X$ and $\sigma : B \rightarrow X$ there exist at most one morphism $\rho \xrightarrow{\varkappa} \sigma$. Indeed, a morphism \varkappa in diagram (0.6) is unique, since the monomorphicity of σ gives the following implication: $\sigma \circ \varkappa = \rho = \sigma \circ \varkappa' \implies \varkappa = \varkappa'$. \square

Remark 0.3. By Example 0.3 this means that in the category $\text{Mono}(X)$ all morphisms are bimorphisms. The connection between the properties of a morphism $\rho \xrightarrow{\varkappa} \sigma$ in $\text{Mono}(X)$ and the properties of the same morphism $\varkappa : A \rightarrow B$ in the initial category \mathbf{K} , is expressed in the following observations:

- every morphism $\rho \xrightarrow{\varkappa} \sigma$ in $\text{Mono}(X)$ is a monomorphism in \mathbf{K} ,
- a morphism $\rho \xrightarrow{\varkappa} \sigma$ in $\text{Mono}(X)$ is an isomorphism in $\text{Mono}(X) \iff \varkappa$ is an isomorphism in \mathbf{K} .

Proof. 1. A morphism \varkappa in (0.6) must be a monomorphism due to Property 1° at the p.7, since $\sigma \circ \varkappa$ is a monomorphism.

2. If a morphism $\varkappa : A \rightarrow B$ in (0.6) is an isomorphism in \mathbf{K} , then we can put $\lambda = \varkappa^{-1} : A \leftarrow B$, and the diagrams

$$\begin{array}{ccc} \begin{array}{c} A \\ \downarrow \lambda \circ \varkappa \\ A \end{array} & \begin{array}{c} \xrightarrow{\rho} \\ \searrow \varkappa \\ \xrightarrow{\sigma} \end{array} & X \\ \begin{array}{c} \downarrow \lambda \\ \downarrow \varkappa \end{array} & \begin{array}{c} \xrightarrow{\rho} \\ \searrow \varkappa \\ \xrightarrow{\sigma} \end{array} & X \end{array} \quad \begin{array}{ccc} \begin{array}{c} B \\ \downarrow \lambda \\ B \end{array} & \begin{array}{c} \xrightarrow{\sigma} \\ \searrow \varkappa \\ \xrightarrow{\rho} \end{array} & X \\ \begin{array}{c} \downarrow \lambda \\ \downarrow \varkappa \end{array} & \begin{array}{c} \xrightarrow{\sigma} \\ \searrow \varkappa \\ \xrightarrow{\rho} \end{array} & X \end{array} \quad (0.7)$$

will be commutative, since ρ and σ are monomorphisms. They mean that the morphisms $\rho \xrightarrow{\varkappa} \sigma$ and $\sigma \xrightarrow{\lambda} \rho$ in $\text{Mono}(X)$ are inverse to each other. Conversely, if morphisms $\rho \xrightarrow{\varkappa} \sigma$ and $\sigma \xrightarrow{\lambda} \rho$ are inverse to each other in $\text{Mono}(X)$, then this means that diagrams (0.7) are commutative. Hence, morphisms \varkappa and λ are inverse to each other in \mathbf{K} , and thus \varkappa must be an isomorphism in \mathbf{K} . \square

It is convenient to introduce a special notation, \rightarrow , for the pre-order in $\text{Mono}(X)$:

$$\rho \rightarrow \sigma \iff \exists \varkappa \in \text{Mor}(\mathbf{K}) \quad \rho = \sigma \circ \varkappa. \quad (0.8)$$

Here the morphism \varkappa , if it exists, must be unique (this follows from the fact that σ is a monomorphism). As a corollary, there is an operation, which to any pair of morphisms $\rho, \sigma \in \text{Mono}(X)$ with the property $\rho \rightarrow \sigma$ assigns the morphism $\varkappa = \varkappa_{\rho}^{\sigma}$ in (0.12):

$$\rho = \sigma \circ \varkappa_{\rho}^{\sigma}. \quad (0.9)$$

If $\rho \rightarrow \sigma \rightarrow \tau$, then the chain

$$\tau \circ \varkappa_\rho^\tau = \rho = \sigma \circ \varkappa_\sigma^\sigma = \tau \circ \varkappa_\sigma^\tau \circ \varkappa_\rho^\sigma,$$

implies, due to monomorphy of τ , the equality

$$\varkappa_\rho^\tau = \varkappa_\sigma^\tau \circ \varkappa_\rho^\sigma. \quad (0.10)$$

- A *system of subobjects* in an object X of a category \mathbf{K} is an arbitrary skeleton S of the category $\mathbf{Mono}(X)$, such that the morphism 1_X belongs to S . In other words, a subclass S in $\mathbf{Mono}(X)$ is a system of subobjects in X , if

(a) the local identity of X belongs to S :

$$1_X \in S,$$

(b) every monomorphism $\mu \in \mathbf{Mono}(X)$ has an isomorphic monomorphism in the class S :

$$\forall \mu \in \mathbf{Mono}(X) \quad \exists \sigma \in S \quad \mu \cong \sigma.$$

(c) in S an isomorphism (in the sense of category $\mathbf{Mono}(X)$) is equivalent to the identity:

$$\forall \sigma, \tau \in S \quad \left(\sigma \cong \tau \iff \sigma = \tau \right)$$

Elements of S are called *subobjects* of X . The class S is endowed with the structure of a full subcategory in $\mathbf{Mono}(X)$.

Theorem 0.3. Any system of subobjects S of an object X is a partially ordered class.

Proof. Let subobjects $\rho \in S$ and $\sigma \in S$ have two mutually inverse morphisms $\varkappa : A \leftarrow B$ and $\lambda : A \rightarrow B$, i.e.

$$\rho = \sigma \circ \varkappa, \quad \sigma = \rho \circ \lambda.$$

Then

$$\rho \circ \lambda \circ \varkappa = \rho = \rho \circ 1_A, \quad \sigma \circ \varkappa \circ \lambda = \sigma = \sigma \circ 1_B,$$

and, since ρ and σ are monomorphisms in \mathbf{K} , one can cancel them:

$$\lambda \circ \varkappa = 1_A, \quad \varkappa \circ \lambda = 1_B,$$

Thus, \varkappa and λ are isomorphisms. We obtain that $\rho \cong \sigma$, and by property (c), $\rho = \sigma$. □

Theorem 0.4. If S is a system of subobjects in X , then for any subobject $\sigma \in S$, $\sigma : Y \rightarrow X$, the class of monomorphisms

$$A = \{ \alpha \in \mathbf{Mono}(Y) : \sigma \circ \alpha \in S \}$$

is a system of subobjects in Y . If in addition S is a set, then A is a set as well.

Proof. 1. Property (a) is obvious: since $\sigma \circ 1_Y = \sigma \in S$, we have that $1_Y \in A$.

2. Property (b). Let $\beta : B \rightarrow Y$ be a monomorphism. The composition $\sigma \circ \beta : B \rightarrow X$ is a monomorphism from $\mathbf{Mono}(X)$, and since S is a system of subobjects in X , there must exist $\tau \in S$ such that

$$\tau \cong \sigma \circ \beta.$$

This means that

$$\tau = \sigma \circ \beta \circ \iota$$

for some isomorphism ι . Now we get that the monomorphism $\alpha = \beta \circ \iota$ is isomorphic to β

$$\alpha \cong \beta$$

and lies in A , since $\sigma \circ \alpha = \tau \in S$.

3. Property (c). Let $\alpha, \beta \in A$ be two isomorphic monomorphisms, i.e.

$$\alpha = \beta \circ \iota$$

for some isomorphism ι . Then, first, the morphisms $\sigma \circ \alpha$ and $\sigma \circ \beta$ are isomorphic as well, since

$$\sigma \circ \alpha = \sigma \circ \beta \circ \iota.$$

And, second, they lay in S , since α and β lay in A . But S satisfies (c), hence the morphisms $\sigma \circ \alpha$ and $\sigma \circ \beta$ coincide:

$$\sigma \circ \alpha = \sigma \circ \beta.$$

In addition σ is a monomorphism, so we have $\alpha = \beta$.

4. It remains to check that if S is a set, then A is a set as well. This follows from the fact that the map $\alpha \in A \mapsto \sigma \circ \alpha \in S$ is injective. Indeed, if for some $\alpha, \alpha' \in A$ we have

$$\sigma \circ \alpha = \sigma \circ \alpha',$$

then, since σ is a monomorphism, we have $\alpha = \alpha'$. □

The category of epimorphisms $\mathbf{Epi}(X)$ and systems of quotient objects. Let X be an object in a category \mathbf{K} . We denote by $\mathbf{Epi}(X)$ the class of all epimorphisms with X as a domain. This is a category where a morphism $\rho \xrightarrow{\varkappa} \sigma$ from an object $\rho \in \mathbf{Epi}(X)$ into an object $\sigma \in \mathbf{Epi}(X)$, i.e. from an epimorphism $\rho : X \rightarrow A$ into an epimorphism $\sigma : X \rightarrow B$, is an arbitrary morphism $\varkappa : A \rightarrow B$ in \mathbf{K} such that the following diagram is commutative

$$\begin{array}{ccc} & & A \\ & \nearrow \rho & \downarrow \varkappa \\ X & & B \\ & \searrow \sigma & \end{array} \quad (0.11)$$

Actually, this diagram in the initial category \mathbf{K} can be considered as a morphism $\rho \xrightarrow{\varkappa} \sigma$ in $\mathbf{Epi}(X)$. A composition of two such morphisms $\rho \xrightarrow{\varkappa} \sigma$ and $\sigma \xrightarrow{\lambda} \tau$, i.e. diagrams

$$\begin{array}{ccc} & & A \\ & \nearrow \rho & \downarrow \varkappa \\ X & & B \\ & \searrow \sigma & \end{array} \quad \begin{array}{ccc} & & B \\ & \nearrow \sigma & \downarrow \lambda \\ X & & C \\ & \searrow \tau & \end{array}$$

is a morphism $\rho \xrightarrow{\lambda \circ \varkappa} \tau$, i.e. a diagram

$$\begin{array}{ccc} & & A \\ & \nearrow \rho & \downarrow \lambda \circ \varkappa \\ X & & C \\ & \searrow \tau & \end{array}$$

One can conceive it as a result of splicing of the initial diagrams along the common edge σ , adding the arrow of composition $\lambda \circ \varkappa$, and then throwing away the vertex B together with all its incidental edges:

$$\begin{array}{ccc} & & A \\ & \nearrow \rho & \downarrow \lambda \circ \varkappa \\ X & \xrightarrow{\sigma} B & \\ & \searrow \tau & \end{array}$$

Of course, local identities in $\mathbf{Epi}(X)$ are diagrams of the form

$$\begin{array}{ccc} & & A \\ & \nearrow \rho & \downarrow 1_A \\ X & & A \\ & \searrow \sigma & \end{array}$$

Remark 0.4. The composition of morphisms in $\mathbf{Epi}(X)$ can be defined in two ways. In our definition this operation is connected with the composition in \mathbf{K} through the following identity:

$$\lambda \circ_{\mathbf{Epi}(X)} \varkappa = \lambda \circ_{\mathbf{K}} \varkappa.$$

By analogy with $\text{Mono}(X)$ the following properties of $\text{Epi}(X)$ are proved.

Theorem 0.5. *For any object X the category $\text{Epi}(X)$ is a graph.*

Remark 0.5. By Example 0.3 this means that *in the category $\text{Epi}(X)$ all the morphisms are bimorphisms*. The connection between the properties of a morphism $\rho \xrightarrow{\varkappa} \sigma$ in $\text{Epi}(X)$ and the properties of the same morphism $\varkappa : A \rightarrow B$ in the initial category \mathbf{K} , is expressed in the following observations:

- every morphism $\rho \xrightarrow{\varkappa} \sigma$ in $\text{Epi}(X)$ is an epimorphism in \mathbf{K} ,
- a morphism $\rho \xrightarrow{\varkappa} \sigma$ in $\text{Epi}(X)$ is an isomorphism in $\text{Epi}(X) \iff \varkappa$ is an isomorphism in \mathbf{K} .

It is convenient to introduce a special notation, \rightarrow , for the pre-order in $\text{Epi}(X)$:

$$\rho \rightarrow \sigma \iff \exists \iota \in \text{Mor}(\mathbf{K}) \quad \sigma = \iota \circ \rho. \quad (0.12)$$

Here the morphism ι , if it exists, must be unique (since ρ is an epimorphism). As a corollary, there is an operation, which to each pair of morphisms $\rho, \sigma \in \text{Epi}(X)$ with the property $\rho \rightarrow \sigma$ assigns the morphism $\iota = \iota_\rho^\sigma$ in (0.12):

$$\sigma = \iota_\rho^\sigma \circ \rho. \quad (0.13)$$

If $\pi \rightarrow \rho \rightarrow \sigma$, then the chain

$$\iota_\pi^\sigma \circ \pi = \sigma = \iota_\rho^\sigma \circ \rho = \iota_\rho^\sigma \circ \iota_\pi^\rho \circ \pi,$$

implies by epimorphy of π the equality

$$\iota_\pi^\sigma = \iota_\rho^\sigma \circ \iota_\pi^\rho. \quad (0.14)$$

- A *system of quotient objects* on an object X in a category \mathbf{K} is an arbitrary skeleton Q of the category $\text{Epi}(X)$, such that 1_X belongs to Q . In other words, a subclass Q in $\text{Epi}(X)$ is called a system of quotient objects on X , if

- (a) the local identity of X belongs to Q :

$$1_X \in Q,$$

- (b) every epimorphism $\varepsilon \in \text{Epi}(X)$ has an isomorphic epimorphism in Q :

$$\forall \varepsilon \in \text{Epi}(X) \quad \exists \pi \in Q \quad \varepsilon \cong \pi,$$

- (c) in Q an isomorphism (in the sense of category $\text{Epi}(X)$) is equivalent to the identity:

$$\forall \pi, \rho \in Q \quad \left(\pi \cong \rho \iff \pi = \rho \right)$$

The elements of the class Q are called *quotient objects* on X . The class Q is endowed with the structure of a full subcategory in $\text{Epi}(X)$.

By analogy with Proposition 0.3 and Proposition 0.4 we have

Proposition 0.3. *Any system Q of quotient objects of an object X is a partially ordered class.*

Proposition 0.4. *If Q is a system of quotient objects of an object X , then for any quotient object $\pi \in Q$, $\pi : X \rightarrow Y$, the class of epimorphisms*

$$A = \{ \alpha \in \text{Epi}(Y) : \alpha \circ \pi \in Q \}$$

is a system of quotient objects on Y . If in addition Q is a set, then A is a set as well.

Limits preserving mono- and epimorphisms. By *covariant system* (respectively, by *contravariant system*) in a category \mathbf{K} over a partially ordered set (I, \leq) we mean arbitrary covariant (respectively, contravariant) functor from I into \mathbf{K} .

Proposition 0.5. *If in a covariant system $\{X^j; \iota_i^j\}$ over a directed set (I, \leq) the morphisms ι_i^j are monomorphisms, then in its projective limit $\{X; \pi^j\}$ the morphisms π^j are monomorphisms as well.*

Proof. Let us assume that I is decreasingly directed. Take an index $k \in I$, and let $Y \xrightarrow{\alpha} X$ and $Y \xrightarrow{\beta} X$ be two colinear morphisms such that

$$\pi^k \circ \alpha = \pi^k \circ \beta.$$

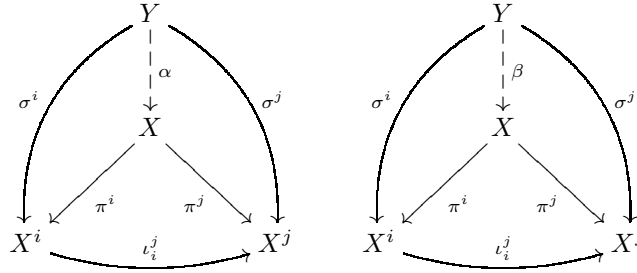
Then for any $j \leq k$ we have:

$$\underbrace{\iota_j^k \circ \pi^j}_{\pi^k} \circ \alpha = \underbrace{\iota_j^k \circ \pi^j}_{\pi^k} \circ \beta.$$

Here ι_j^k is a monomorphism, so we can cancel it:

$$\pi^j \circ \alpha = \pi^j \circ \beta, \quad j \leq k.$$

Set $\sigma^j = \pi^j \circ \alpha = \pi^j \circ \beta$, then morphisms $Y \xrightarrow{\alpha} X$ and $Y \xrightarrow{\beta} X$ generate the same cone of the covariant system $\{X^j; \iota_i^j; i \leq j \leq k\}$:



(the projective limit of a covariant system over a cofinal interval $\{j \in I : j \leq k\}$ is the same as over I , so we substitute X into this place). This implies that α and β coincide by the uniqueness of the corresponding arrow in the definition of projective limit:

$$\alpha = \beta$$

□

The dual proposition is the following:

Proposition 0.6. *If in a covariant system $\{X^j; \iota_i^j\}$ over a directed set (I, \leq) the morphisms ι_i^j are epimorphisms, then in its injective limit $\{X; \rho_i\}$ the morphisms ρ_i are epimorphisms as well.*

Remark 0.6. If the set of indices I is not directed, then the projective (injective) limit of a covariant system of monomorphisms (epimorphisms) over it is not necessarily a cone of monomorphisms (epimorphisms). For example if the order in I is discrete, i.e. $i \leq j \Leftrightarrow i = j$, then the projective limit of any covariant system $\{X^i; \iota_i^j\}$ over I is the direct product $\prod_{i \in I} X^i$, where the projections

$$\prod_{i \in I} X^i \xrightarrow{\pi^k} X^k$$

as a rule are not monomorphisms (although the initial morphisms $\iota_i^i = 1_{X^i}$ are monomorphisms). Similarly, injective limit of $\{X^i; \iota_i^j\}$ is a coproduct $\coprod_{i \in I} X_i$, and the corresponding injections

$$X_k \xrightarrow{\rho_k} \coprod_{i \in I} X_i$$

as a rule are not epimorphisms here (although $\iota_i^i = 1_{X_i}$ are epimorphisms).

(c) Immediate monomorphisms and immediate epimorphisms.

- We call a *factorization* of a morphism $X \xrightarrow{\varphi} Y$ any its representation as a composition of epimorphism and a monomorphism, i.e. any commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ & \searrow \varepsilon & \nearrow \mu \\ & M & \end{array} \quad (0.15)$$

where ε is an epimorphism, and μ a monomorphism.

- A monomorphism $\mu : X \rightarrow Y$ is said to be *immediate*, if in any its factorization $\mu = \mu' \circ \varepsilon$ the epimorphism ε is automatically an isomorphism. Note that for a monomorphism μ in any its factorization $\mu = \mu' \circ \varepsilon$ the epimorphism ε is automatically a bimorphism. As a corollary, the condition of being immediate monomorphism for μ is equivalent to the requirement that in any decomposition $\mu = \mu' \circ \varepsilon$, where ε is a bimorphism, and μ' a monomorphism, the morphism ε must be an isomorphism. It is natural to call a monomorphism μ' in the factorization $\mu = \mu' \circ \varepsilon$ a *mediator* of the monomorphism μ , then the epithet “immediate” for μ will mean that there are no non-trivial mediators for μ (i.e. mediators, which are not isomorphic to μ in $\text{Mono}(Y)$).
- An epimorphism $\varepsilon : X \rightarrow Y$ is said to be *immediate*, if in any its factorization $\varepsilon = \mu \circ \varepsilon'$ the monomorphism μ is automatically an isomorphism. Note that for an epimorphism ε in any its factorization $\varepsilon = \mu \circ \varepsilon'$ the monomorphism μ is automatically a bimorphism. As a corollary, the condition of being immediate epimorphism for ε is equivalent to the requirement that in any decomposition $\varepsilon = \mu \circ \varepsilon'$, where μ is a bimorphism, and ε' an epimorphism, the morphism μ must be an isomorphism. It is natural to call an epimorphism ε' in the factorization $\varepsilon = \mu \circ \varepsilon'$ a *mediator* of the epimorphism ε , then the epithet “immediate” for ε will mean that there are no non-trivial mediators for ε (i.e. mediators, which are not isomorphic to ε in $\text{Epi}(X)$).

Remark 0.7. If in the definition of the immediate monomorphism we omit the requirement that the morphism μ' in the representation $\mu = \mu' \circ \varepsilon$ is a monomorphism (i.e. if we claim only that the epimorphy of ε must imply its isomorphy), then we obtain exactly the definition of the so-called *extremal monomorphism*. Similarly, if in the definition of the immediate epimorphism we omit the requirement that the morphism ε' in the representation $\varepsilon = \mu \circ \varepsilon'$ is an epimorphism (i.e. if we claim only that the monomorphy of μ implies its isomorphy), then we obtain the definition of the *extremal epimorphism* [5, Definition 4.3.2]. Certainly, each extremal monomorphism (respectively, extremal epimorphism) is an immediate monomorphism (respectively, immediate epimorphism). But the reverse implication is not true, and the following example shows this⁴. Consider a monoid $\langle a, b, c \mid ac = bc \rangle$ (generated by three elements a, b, c with the equality $ac = bc$) as a category with the one object. In this category

- 1) the morphisms a, b, c are monomorphisms (since they can be canceled in the equalities like $a \cdot P = a \cdot Q$),
- 2) the morphisms a, b are epimorphisms (since they can be canceled in the equalities like $P \cdot a = Q \cdot a$),
- 3) the morphism c is not an epimorphism (since it cannot be canceled in the equality $a \cdot c = b \cdot c$),
- 4) the morphism $ac = bc$ is
 - a monomorphism (since it can be canceled in the equalities like $ac \cdot P = ac \cdot Q$),
 - an epimorphism (since it can be canceled in the equalities like $P \cdot ac = Q \cdot ac$),
 - an immediate epimorphism (since there is only one possibility to write it in the form $(\text{mono}) \circ (\text{epi})$, namely, $ac = 1 \cdot (ac)$, and then the first morphism in this decomposition, i.e. 1, is an isomorphism),
 - but not an extremal epimorphism (since it can be written in the form $(\text{mono}) \circ (\dots)$, namely, $ac = a \cdot c$, where the first morphism, i.e. a , is not an isomorphism).

In addition, the morphism $acac$ is not an immediate epimorphism, since it can be represented as

$$acac = \underbrace{(ac)}_{\text{Mono}} \cdot \underbrace{(ac)}_{\text{Epi}}$$

where the first morphism is not an isomorphism. This shows that *a composition of two immediate monomorphisms (respectively, of two immediate epimorphisms) is not necessarily an immediate monomorphism (respectively, an immediate epimorphism).*

Properties of immediate mono- and epimorphisms:

- 1°. If $\varphi \circ \mu$ is an immediate monomorphism, then μ is an immediate monomorphism as well.
- 2°. If μ is an immediate monomorphism, and at the same time an epimorphism, then μ is an isomorphism.
- 3°. If $\varepsilon \circ \varphi$ is an immediate epimorphism, then ε is an immediate epimorphism as well.
- 4°. If ε is an immediate epimorphism, and at the same time a monomorphism, then ε is an isomorphism.

⁴This example was suggested to the author by B. V. Novikov.

(d) Strong monomorphisms and strong epimorphisms.

The following two definitions are due to M. Sh. Tsalenko and E. G. Shulgeifer [38, Chapter 1 §7] and F. Borceux [5, 4.3].

- A monomorphism $C \xrightarrow{\mu} D$ is said to be *strong*, if for any epimorphism $A \xrightarrow{\varepsilon} B$ and for any morphisms $A \xrightarrow{\alpha} C$ and $B \xrightarrow{\beta} D$ such that $\beta \circ \varepsilon = \mu \circ \alpha$ there exists (the only possible) morphism $B \xrightarrow{\delta} C$, such that the following diagram will be commutative:

$$\begin{array}{ccc}
 A & \xrightarrow{\varepsilon} & B \\
 \alpha \downarrow & \delta \swarrow & \downarrow \beta \\
 C & \xrightarrow{\mu} & D
 \end{array} \quad (0.16)$$

- Dually, an epimorphism $A \xrightarrow{\varepsilon} B$ is said to be *strong*, if for any monomorphism $C \xrightarrow{\mu} D$ and for any morphisms $A \xrightarrow{\alpha} C$ and $B \xrightarrow{\beta} D$ such that $\beta \circ \varepsilon = \mu \circ \alpha$ there exists (the only possible) morphism $B \xrightarrow{\delta} C$, such that diagram (0.16) is commutative.

Remark 0.8. The uniqueness of δ follows from monomorphicity of μ (or from epimorphicity of ε): if δ' is another morphism with the same property, then

$$\mu \circ \delta = \beta \circ \mu \circ \delta' \implies \delta = \delta'.$$

Besides this, the commutativity of the upper triangle in (0.16) imply the commutativity of the lower one, and vice versa. For example,

$$\alpha = \delta \circ \varepsilon \implies \beta \circ \underset{\text{Epi}}{\varepsilon} = \mu \circ \alpha = \mu \circ \delta \circ \underset{\text{Epi}}{\varepsilon} \implies \beta = \mu \circ \delta \quad (0.17)$$

The following propositions are proved in [5, Proposition 4.3.6]:

Proposition 0.7. *A composition of two strong monomorphisms (respectively, of two strong epimorphisms) is a strong monomorphism (respectively, a strong epimorphism).*

Properties of strong mono- and epimorphisms:

- 1°. If $\varphi \circ \mu$ is a strong monomorphism, then μ is a strong monomorphism as well.
- 2°. Every strong monomorphism μ is an immediate monomorphism.
- 3°. If $\varepsilon \circ \varphi$ is a strong epimorphism, then ε is a strong epimorphism as well.
- 4°. Every strong epimorphism ε is an immediate epimorphism.

The category of strong monomorphisms $\mathbf{SMono}(X)$ and systems of strong subobjects. As we already told on page 7, we omit the proofs of the propositions of this and the next subsection, since they copy the analogous statements for mono/epimorphisms.

- Let X be an object in a category \mathbf{K} . We denote by $\mathbf{SMono}(X)$ the class of all strong monomorphisms having X as a range. This is a category where a morphism from an object $\rho \in \mathbf{SMono}(X)$ into an object $\sigma \in \mathbf{SMono}(X)$, i.e. a strong monomorphism $\rho : A \rightarrow X$ into a strong monomorphism $\sigma : B \rightarrow X$ as any morphism $\varkappa : A \rightarrow B$ such that the following diagram is commutative

$$\begin{array}{ccc}
 A & & X \\
 \varkappa \downarrow & \rho \searrow & \\
 B & \xrightarrow{\sigma} & X
 \end{array} \quad (0.18)$$

Obviously, $\mathbf{SMono}(X)$ is a (full, due to 1° above) subcategory in $\mathbf{Mono}(X)$.

Theorem 0.6. *For any object X the category $\mathbf{SMono}(X)$ is a graph.*

Remark 0.9. By Example 0.3 in $\mathbf{SMono}(X)$ all morphisms are bimorphisms. The connection between the properties of a morphism $\rho \xrightarrow{\varkappa} \sigma$ in $\mathbf{SMono}(X)$ and the properties of the same morphism $\varkappa : A \rightarrow B$ in the initial category \mathbf{K} , is expressed in the following observations:

- every morphism $\rho \xrightarrow{\varkappa} \sigma$ in $\mathbf{SMono}(X)$ is a strong monomorphism in \mathbf{K} ,
- a morphism $\rho \xrightarrow{\varkappa} \sigma$ in $\mathbf{SMono}(X)$ is an isomorphism in $\mathbf{SMono}(X) \iff \varkappa$ is an isomorphism in \mathbf{K} .
- A system of strong subobjects in an object X of a category \mathbf{K} is an arbitrary skeleton S of the category $\mathbf{SMono}(X)$, such that the local identity 1_X belongs to S . In other words, a subclass S in $\mathbf{SMono}(X)$ is called a system of strong subobjects in X , if
 - (a) the local identity of X belongs to S :

$$1_X \in S,$$

- (b) every strong monomorphism $\mu \in \mathbf{SMono}(X)$ has an isomorphic monomorphism from S :

$$\forall \mu \in \mathbf{Mono}(X) \quad \exists \sigma \in S \quad \mu \cong \sigma.$$

- (c) in S an isomorphism (in the sense of category $\mathbf{SMono}(X)$) is equivalent to identity:

$$\forall \sigma, \tau \in S \quad \left(\sigma \cong \tau \iff \sigma = \tau \right)$$

The elements of S are called *strong subobjects* in X . The class S is endowed with the structure of a full subcategory in $\mathbf{SMono}(X)$.

Proposition 0.8. *Any system of strong subobjects S of an object X is always a partially ordered class.*

Proposition 0.9. *If S is a system of strong subobjects in an object X , then for any strong subobject $\sigma \in S$, $\sigma : Y \rightarrow X$, the class of strong monomorphisms*

$$A = \{ \alpha \in \mathbf{SMono}(Y) : \sigma \circ \alpha \in S \}$$

is a system of strong subobjects in the object Y . If in addition S is a set, then A is a set as well.

The category of strong epimorphisms $\mathbf{SEpi}(X)$ and systems of strong quotient objects.

- Let X be an object in a category \mathbf{K} . We denote by $\mathbf{SEpi}(X)$ the class of all strong epimorphisms with X as a domain. This is a category where a morphism from an object $\rho \in \mathbf{SEpi}(X)$ into an object $\sigma \in \mathbf{SEpi}(X)$, i.e. an strong epimorphism $\rho : X \rightarrow A$ into a strong epimorphism $\sigma : X \rightarrow B$, is an arbitrary morphism $\varkappa : A \rightarrow B$ such that the following diagram is commutative

$$\begin{array}{ccc} & A & \\ \rho \nearrow & & \downarrow \varkappa \\ X & & B \\ \sigma \searrow & & \end{array} \quad (0.19)$$

Obviously, $\mathbf{SEpi}(X)$ is a (full, due to 3° above) subcategory in $\mathbf{Epi}(X)$.

Theorem 0.7. *For any X the category $\mathbf{SEpi}(X)$ is a graph.*

Remark 0.10. By Example 0.3 in $\mathbf{SEpi}(X)$ all morphisms are bimorphisms. The connection between the properties of a morphism $\rho \xrightarrow{\varkappa} \sigma$ in $\mathbf{SEpi}(X)$ and the properties of the same morphism $\varkappa : A \rightarrow B$ in the initial category \mathbf{K} , is expressed in the following observations:

- every morphism $\rho \xrightarrow{\varkappa} \sigma$ in $\mathbf{SEpi}(X)$ is a strong epimorphism in \mathbf{K} ,
- a morphism $\rho \xrightarrow{\varkappa} \sigma$ in $\mathbf{SEpi}(X)$ is an isomorphism in $\mathbf{SEpi}(X) \iff \varkappa$ is an isomorphism in \mathbf{K} .
- A system of strong quotient objects on an object X of a category \mathbf{K} is an arbitrary skeleton Q of the category $\mathbf{SEpi}(X)$, such that the local identity 1_X belongs to Q . In other words, Q is a subclass in $\mathbf{SEpi}(X)$ such that

(a) the local identity of X belongs to Q :

$$1_X \in Q,$$

(b) every strong epimorphism $\varepsilon \in \mathbf{SEpi}(X)$ has an isomorphic epimorphism in Q :

$$\forall \varepsilon \in \mathbf{Epi}(X) \quad \exists \pi \in Q \quad \varepsilon \cong \pi,$$

(c) in Q an isomorphism (in the sense of the category $\mathbf{SEpi}(X)$) is equivalent to the identity:

$$\forall \pi, \rho \in Q \quad \left(\pi \cong \rho \iff \pi = \rho \right)$$

Elements of Q are called *strong quotient objects* on X . The class Q is endowed with the structure of a full subcategory in $\mathbf{SEpi}(X)$.

Proposition 0.10. *Any system Q of strong quotient objects on an object X is always a partially ordered class.*

Proposition 0.11. *If Q is a system of strong quotient objects for an object X , then for any strong quotient object $\pi \in Q$, $\pi : X \rightarrow Y$, the class of epimorphisms*

$$A = \{\alpha \in \mathbf{SEpi}(Y) : \alpha \circ \pi \in Q\}$$

is a system of strong quotient objects on the object Y . If in addition Q is a set, then A is a set as well.

Limits preserving strong mono- and epimorphisms.

Proposition 0.12. *If in a covariant system $\{X^j; \iota_i^j\}$ over a decreasingly directed set (I, \leq) the morphisms ι_i^j are strong monomorphisms, then in its projective limit $\{X; \pi^j\}$ the morphisms π^j are strong monomorphisms as well.*

Proof. Take an index $k \in I$. By Proposition 0.5, π^k is a monomorphism, so we need only to show that it is strong. Consider a diagram

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon} & B \\ \alpha \downarrow & & \downarrow \beta \\ X & \xrightarrow{\pi^k} & X^k \end{array}$$

where ε is an epimorphism. For any index $j \leq k$ we can construct a diagram

$$\begin{array}{ccccc} A & & \xrightarrow{\varepsilon} & & B \\ & \searrow \pi^j \circ \alpha & & & \downarrow \beta \\ \alpha \downarrow & & X^j & & \\ & \nearrow \pi^j & \searrow \iota_j^k & & \\ X & \xrightarrow{\pi^k} & X^k \end{array}$$

and consider the following fragment:

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon} & B \\ \pi^j \circ \alpha \searrow & & \downarrow \beta \\ & X^j & \\ & \searrow \iota_j^k & \\ & & X^k \end{array}$$

Since ε is an epimorphism, and ι_j^k is a strong monomorphism, there exists (a unique) morphism δ^j such that the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon} & B \\ \pi^j \circ \alpha \searrow & \delta^j \swarrow & \downarrow \beta \\ & X^j & \\ & \searrow \iota_j^k & \\ & & X^k \end{array}$$

In particular,

$$\iota_j^k \circ \delta^j = \beta, \quad j \leq k$$

As a corollary, if we take a new index $i \leq j$, then for the arising morphisms δ^j and δ^i we get

$$\iota_j^k \circ \delta^j = \beta = \iota_k^i \circ \delta^i = \iota_j^k \circ \iota_i^j \circ \delta^i.$$

Here ι_j^k is a monomorphism, so we can cancel it:

$$\delta^j = \iota_i^j \circ \delta^i.$$

Thus for any $i \leq j \leq k$ the following diagram is commutative:

$$\begin{array}{ccc} & & B \\ & \swarrow \delta^i & \searrow \delta^j \\ X^i & & X^j \\ & \searrow \iota_i^j & \swarrow \end{array}$$

(for $j = k$ we have $\delta^k = \beta$).

This means that the system of morphisms $\{\delta^j : B \rightarrow X^j; j \leq k\}$ is a projective cone of a covariant system $\{\iota_i^j : X^i \rightarrow X^j; i \leq j \leq k\}$. Hence, there exists a unique morphism $\delta : B \rightarrow X$ such that all the following diagrams are commutative:

$$\begin{array}{ccc} & B & \\ \delta \swarrow & & \searrow \delta^j \\ X & \xrightarrow{\pi^j} & X^j \end{array}$$

(the limit along a cofinal interval $\{j \in I : j \leq k\}$ coincides with the limit along I).

In particular, for $j = k$ we get a commutative diagram

$$\begin{array}{ccc} & B & \\ \delta \swarrow & & \searrow \beta \\ X & \xrightarrow{\pi^k} & X^k \end{array}$$

It implies the following chain:

$$\beta = \pi^k \circ \delta \implies \underbrace{\pi^k}_{\text{Mono}} \circ \alpha = \beta \circ \varepsilon = \underbrace{\pi^k}_{\text{Mono}} \circ \delta \circ \varepsilon \implies \alpha = \delta \circ \varepsilon$$

Thus, the following square is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon} & B \\ \alpha \downarrow & \delta \swarrow & \searrow \beta \\ X & \xrightarrow{\pi^k} & X^k \end{array}$$

□

The dual proposition is the following:

Proposition 0.13. *If in a covariant system $\{X^j; \iota_i^j\}$ over an increasingly directed set (I, \leq) the morphisms ι_i^j are strong epimorphisms, then in its injective limit $\{X; \rho_i\}$ the morphisms ρ_i are strong epimorphisms as well.*

(e) Well-powered and co-well-powered categories

Later (see page 60) we will use the following definition.

- A category \mathbf{K} is said to be
 - *well-powered* (respectively, *strongly well-powered*), if every object X in \mathbf{K} has a system of subobjects (respectively, strong subobjects) S , which is a set (i.e. not just a class); clearly, a reformulation of this condition is a claim, that all the categories $\mathbf{Mono}(X)$ (respectively, all the categories $\mathbf{SMono}(X)$) are skeletally small;
 - *co-well-powered* (respectively, *strongly co-well-powered*), if every object X in \mathbf{K} has a system of quotient objects (respectively, strong quotient objects) Q , which is a set; again, a reformulation is a claim that all the categories $\mathbf{Epi}(X)$ (respectively, all the categories $\mathbf{SEpi}(X)$) are skeletally small.

Example 0.4. The standard categories frequently used as examples, like the category of sets, groups, vector spaces, algebras (over a given field), topological spaces, topological vector spaces, topological algebras, etc., are, obviously, well-powered (and as a corollary, strongly well-powered). Some of them, like the category of vector spaces are again, obviously, co-well-powered as well. But for some others, the property of being co-well-powered is proved in an amazingly complicated way (see [1]). However, the weaker property of being strongly co-well-powered is verified much easier.

§ 1 Envelope and imprint

(a) Envelope

Envelope in a class of morphisms with respect to a class of morphisms. Suppose we have:

- a category \mathbf{K} called an *enveloping category*,
- a category \mathbf{T} called an *attracting category*,
- a covariant functor $F : \mathbf{T} \rightarrow \mathbf{K}$,
- two classes Ω and Φ of morphisms in \mathbf{K} , taking values in objects of the class $F(\mathbf{T})$, and Ω is called the *class of realizing morphisms*, and Φ the *class of test morphisms*.

Then

- For given objects $X \in \mathbf{Ob}(\mathbf{K})$ and $X' \in \mathbf{Ob}(\mathbf{T})$ a morphism $\sigma : X \rightarrow F(X')$ is called an *extension of object $X \in \mathbf{K}$ over the category \mathbf{T} in the class of morphisms Ω with respect to the class of morphisms Φ* , if $\sigma \in \Omega$, and for each morphism $\varphi : X' \rightarrow B$ from the class Φ there exists a unique morphism $\varphi' : X' \rightarrow B$ in the category \mathbf{T} such that the following diagram is commutative:

$$\begin{array}{ccc}
 & X & \\
 \Omega \ni \sigma \swarrow & & \searrow \varphi \in \Phi \\
 F(X') & \xrightarrow{\quad F(\varphi') \quad} & F(B)
 \end{array} \tag{1.1}$$

- An extension $\rho : X \rightarrow F(E)$ of an object $X \in \mathbf{K}$ over a category \mathbf{T} in the class of morphisms Ω with respect to the class of morphisms Φ is called an *envelope of the object $X \in \mathbf{K}$ over the category \mathbf{T} in the class of morphisms Ω with respect to the class of morphisms Φ* , and we write in this case

$$\rho = \text{env}_{\Phi}^{\Omega} X, \tag{1.2}$$

if for each extension $\sigma : X \rightarrow F(X')$ (of the object $X \in \mathbf{K}$ over the category \mathbf{T} in the class of morphisms Ω with respect to the class of morphisms Φ) there exists a unique morphism $v : X' \rightarrow E$ in \mathbf{T} such that the following diagram is commutative

$$\begin{array}{ccc}
 & X & \\
 \sigma \swarrow & & \searrow \rho \\
 F(X') & \xrightarrow{\quad F(v) \quad} & F(E)
 \end{array} \tag{1.3}$$

The object E will also be called an *envelope* of X (over the category \mathbf{T} in the class of morphisms Ω with respect to the class of morphisms Φ), and we write also

$$E = \mathbf{Env}_{\Phi}^{\Omega} X. \quad (1.4)$$

Further we are almost exclusively interested in the case when $\mathbf{T} = \mathbf{K}$, and $F : \mathbf{K} \rightarrow \mathbf{K}$ is the identity functor. In doing this we will omit in the terminology the mentioning of the category \mathbf{T} , so we'll be speaking about *extensions* and *envelopes in the class of morphisms Ω with respect to the class of morphisms Φ* . The diagrams (1.1) and (1.3) will turn respectively into diagrams

$$\begin{array}{ccc} & X & \\ \Omega \ni \sigma \swarrow & & \searrow \forall \varphi \in \Phi \\ X' & \dashrightarrow & B \\ & \exists! \varphi' & \end{array} \quad (1.5)$$

and

$$\begin{array}{ccc} & X & \\ \forall \sigma \swarrow & & \searrow \rho \\ X' & \dashrightarrow & E \\ & \exists! v & \end{array} \quad (1.6)$$

Remark 1.1. Clearly, the object $\mathbf{Env}_{\Phi}^{\Omega} X$ (if it exists) is defined up to an isomorphism. If we want to have a map $X \mapsto \mathbf{Env}_{\Phi}^{\Omega} X$, then (providing that the envelope always exists) the sufficient condition for existence of this map is the skeletally smallness of the category \mathbf{K} . In the important special case when Ω coincides with the class \mathbf{Epi} of all epimorphisms of the category \mathbf{K} , the sufficient condition for existence of the map $X \mapsto \mathbf{Env}_{\Phi}^{\Omega} X$ (again, providing that the envelope always exists) is that the category \mathbf{K} is co-well-powered (since in this case the object $\mathbf{Env}_{\Phi}^{\mathbf{Epi}} X$ can be chosen by the axiom of choice as a concrete quotient object of X).

Remark 1.2. If $\Omega = \emptyset$, then, of course neither extensions, nor envelopes in Ω exist. So this construction can be interesting only when Ω is a non-empty class. The following two situations will be of special interest

- $\Omega = \mathbf{Epi}(\mathbf{K})$ (i.e. Ω coincides with the class of all epimorphisms in the category \mathbf{K}), then we will use the following notations

$$\mathbf{env}_{\Phi}^{\mathbf{Epi}} X := \mathbf{env}_{\Phi}^{\mathbf{Epi}(\mathbf{K})} X, \quad \mathbf{Env}_{\Phi}^{\mathbf{Epi}} X := \mathbf{Env}_{\Phi}^{\mathbf{Epi}(\mathbf{K})} X. \quad (1.7)$$

- $\Omega = \mathbf{Mor}(\mathbf{K})$ (i.e. Ω coincides with the class of all morphisms in the category \mathbf{K}), in this case it is convenient to omit any mentioning about Ω in the formulations and notations, so we will be speaking about the *envelope of object $X \in \mathbf{K}$ in the category \mathbf{K} with respect to the class of morphisms Φ* , and the notations will be simplified as follows:

$$\mathbf{env}_{\Phi} X := \mathbf{env}_{\Phi}^{\mathbf{Mor}(\mathbf{K})} X, \quad \mathbf{Env}_{\Phi} X := \mathbf{Env}_{\Phi}^{\mathbf{Mor}(\mathbf{K})} X. \quad (1.8)$$

Remark 1.3. Another degenerate, but this time an informative case is when $\Phi = \emptyset$. What is essential for a given object X , Φ does not contain morphisms going from X :

$$\Phi_X = \{\varphi \in \Phi : \mathbf{Dom}(\varphi) = X\} = \emptyset.$$

Then, obviously, any morphism $\sigma \in \Omega$, going from X , $\sigma : X \rightarrow X'$, is an extension for X (in the class Ω with respect to the class \emptyset). If in addition $\Omega = \mathbf{Epi}$, then the envelope for X is the terminal object in the category $\mathbf{Epi}(X)$ (if it exists). This can be depicted by the formula

$$\mathbf{Env}_{\emptyset}^{\Omega} X = \max \mathbf{Epi}(X).$$

In particular, if \mathbf{K} is a category with zero 0 , and Ω contains all morphisms going to 0 , then the envelope of any object with respect to the empty class of morphisms is 0 :

$$\mathbf{Env}_{\emptyset}^{\Omega} X = 0.$$

Remark 1.4. Another extreme situation is when $\Phi = \text{Mor}(\mathbb{K})$. For a given object X the essential thing here is that the class Φ contains the local identity of X :

$$1_X \in \Phi.$$

For any extension σ the diagram

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X' \\ & \searrow 1_X & \swarrow \text{dotted} \\ & X & \end{array}$$

implies that σ must be a co-retraction (moreover, the dotted arrow here must be unique). In the special case of $\Omega = \text{Epi}$ this is possible only if σ is an isomorphism. As a corollary, in this case the envelope of X coincides with X (up to an isomorphism):

$$\text{Env}_{\text{Mor}(\mathbb{K})}^{\text{Epi}} X = X.$$

Properties of envelopes:

1°. Suppose that $\Sigma \subseteq \Omega$, then for any object X and for any class of morphisms Φ

- (a) each extension $\sigma : X \rightarrow X'$ in Σ with respect to Φ is an extension in Ω with respect to Φ ,
- (b) if there are envelopes $\text{env}_{\Phi}^{\Sigma} X$ and $\text{env}_{\Phi}^{\Omega} X$, then there is a unique morphism $\rho : \text{Env}_{\Phi}^{\Sigma} X \rightarrow \text{Env}_{\Phi}^{\Omega} X$ such that the following diagram is commutative:

$$\begin{array}{ccc} & X & \\ \text{env}_{\Phi}^{\Sigma} X \swarrow & & \searrow \text{env}_{\Phi}^{\Omega} X \\ \text{Env}_{\Phi}^{\Sigma} X & \xrightarrow{\rho} & \text{Env}_{\Phi}^{\Omega} X \end{array} \quad (1.9)$$

- (c) if there is an envelope $\text{env}_{\Phi}^{\Omega} X$ (in a wider class), and it lies in Σ (i.e. in a narrower class),

$$\text{env}_{\Phi}^{\Omega} X \in \Sigma,$$

then it is an envelope $\text{env}_{\Phi}^{\Sigma} X$ (in a narrower class):

$$\text{env}_{\Phi}^{\Omega} X = \text{env}_{\Phi}^{\Sigma} X.$$

2°. Suppose $\Psi \subseteq \Phi$, then for any object X and for any class of morphisms Ω

- (a) each extension $\sigma : X \rightarrow X'$ in Ω with respect to Φ is an extension in Ω with respect to Ψ ,
- (b) if there are envelopes $\text{env}_{\Psi}^{\Omega} X$ and $\text{env}_{\Phi}^{\Omega} X$, then there is a unique morphism $\alpha : \text{Env}_{\Psi}^{\Omega} X \leftarrow \text{Env}_{\Phi}^{\Omega} X$ such that the following diagram is commutative:

$$\begin{array}{ccc} & X & \\ \text{env}_{\Psi}^{\Omega} X \swarrow & & \searrow \text{env}_{\Phi}^{\Omega} X \\ \text{Env}_{\Psi}^{\Omega} X & \xleftarrow{\alpha} & \text{Env}_{\Phi}^{\Omega} X \end{array} \quad (1.10)$$

3°. Suppose that $\Phi \subseteq \text{Mor}(\mathbb{K}) \circ \Psi$ (i.e. each morphism $\varphi \in \Phi$ can be represented as a composition $\varphi = \chi \circ \psi$, where $\psi \in \Psi$), then for any object X and for any class of morphisms Ω

- (a) if an extension $\sigma : X \rightarrow X'$ in Ω with respect to Ψ is at the same time an epimorphism in \mathbb{K} , then it is an extension in Ω with respect to Φ ,
- (b) if there are envelopes $\text{env}_{\Psi}^{\Omega} X$ and $\text{env}_{\Phi}^{\Omega} X$, and $\text{env}_{\Psi}^{\Omega} X$ is at the same time an epimorphism in \mathbb{K} , then there exists a unique morphism $\beta : \text{Env}_{\Psi}^{\Omega} X \rightarrow \text{Env}_{\Phi}^{\Omega} X$ such that the following diagram is commutative:

$$\begin{array}{ccc} & X & \\ \text{env}_{\Psi}^{\Omega} X \swarrow & & \searrow \text{env}_{\Phi}^{\Omega} X \\ \text{Env}_{\Psi}^{\Omega} X & \xrightarrow{\beta} & \text{Env}_{\Phi}^{\Omega} X \end{array} \quad (1.11)$$

4°. Suppose that Ω and Φ are some classes of morphisms, and $\varepsilon : X \rightarrow Y$ an epimorphism in \mathbf{K} such that the following three conditions are fulfilled:

- (a) there exists an envelope $\text{env}_{\Phi \circ \varepsilon}^\Omega X$ with respect to the class of morphisms $\Phi \circ \varepsilon = \{\varphi \circ \varepsilon; \varphi \in \Phi\}$,
- (b) there exists an envelope $\text{env}_\Phi^\Omega Y$,
- (c) the composition $\text{env}_\Phi^\Omega Y \circ \varepsilon$ belongs to Ω .

Then there exists a unique morphism $v : \text{Env}_{\Phi \circ \varepsilon}^\Omega X \leftarrow \text{Env}_\Phi^\Omega Y$ such that the following diagram is commutative:

$$\begin{array}{ccc}
 X & \xrightarrow{\varepsilon} & Y \\
 \text{env}_{\Phi \circ \varepsilon}^\Omega X \downarrow & \searrow \text{env}_\Phi^\Omega Y \circ \varepsilon & \downarrow \text{env}_\Phi^\Omega Y \\
 \text{Env}_{\Phi \circ \varepsilon}^\Omega X & \xleftarrow{v} & \text{Env}_\Phi^\Omega Y
 \end{array} \quad (1.12)$$

Proof. 1. If the morphism σ satisfies (1.5) with Σ instead of Ω , then σ satisfies the initial condition (1.5), since $\Sigma \subseteq \Omega$. This proves (a). From this we have also that $\text{env}_\Sigma^\Omega X$ is an extension in Ω with respect to Φ , so there must exist a unique dotted arrow in (1.9). This means that (b) is also true. Finally, if there exists an envelope $\text{env}_\Phi^\Omega X$ (in a wider class), and it lies in Σ (in a narrower class), $\text{env}_\Phi^\Omega X \in \Sigma$, then $\text{env}_\Phi^\Omega X$ is an extension in Σ . On the other hand any other extension $\sigma : X \rightarrow X'$ in Σ is an extension in Ω due to the property (a) which we have just proved, hence there is a unique morphism v into the envelope in Ω :

$$\begin{array}{ccc}
 & X & \\
 \sigma \swarrow & & \searrow \text{env}_\Phi^\Omega X \\
 X' & \xleftarrow{v} & \text{Env}_\Phi^\Omega X
 \end{array}$$

This proves that $\text{env}_\Phi^\Omega X$ is an envelope in Σ , and we have proved (c).

2. Suppose that $\Psi \subseteq \Phi$. Then (a) is obvious: each extension $\sigma : X \rightarrow X'$ with respect to Φ is an extension with respect to the narrower class Ψ . For (b) we have: since $\text{env}_\Phi^\Omega X$ is an extension with respect to Φ , it must be an extension with respect to the narrower class Ψ , so there exists a unique morphism from $\text{Env}_\Phi^\Omega X$ into the envelope $\text{Env}_\Psi^\Omega X$ with respect to Ψ such that (1.10) is commutative.

3. Suppose that $\Phi \subseteq \text{Mor}(\mathbf{K}) \circ \Psi$. For (a) our reasoning will be illustrated by the following diagram:

$$\begin{array}{ccc}
 X & \xrightarrow{\sigma} & X' \\
 \psi \searrow & & \swarrow \psi' \\
 & Y & \\
 \varphi \searrow & \downarrow \chi & \swarrow \varphi' \\
 & B &
 \end{array}$$

If $\sigma : X \rightarrow X'$ is an extension of X in Ω with respect to Ψ , then for any morphism $\varphi \in \Phi$, $\varphi : X \rightarrow B$, we take a decomposition $\varphi = \chi \circ \psi$, where $\psi \in \Psi$. Since σ is an extension of X in Ω with respect to Ψ , there is a morphism ψ' such that $\psi = \psi' \circ \sigma$. After that we put $\varphi' = \chi \circ \psi'$, and this will be a morphism such that

$$\varphi = \chi \circ \psi = \chi \circ \psi' \circ \sigma = \varphi' \circ \sigma.$$

The uniqueness of φ' follows from the epimorphy of $\sigma \in \Omega$, and thus σ is an extension of X in Ω with respect to Φ . Once (a) is proved, (b) becomes its corollary: the morphism $\text{env}_\Psi^\Omega X : X \rightarrow \text{Env}_\Psi^\Omega X$ is an extension of X in Ω with respect to Ψ , hence, by (a), with respect to Φ as well. So there must exist a morphism β from $\text{Env}_\Psi^\Omega X$ into the envelope $\text{Env}_\Phi^\Omega X$ with respect to Φ such that (1.11) is commutative.

4. For any morphism $\varphi : Y \rightarrow B$ lying in Φ we have the following diagram:

$$\begin{array}{ccc}
 X & \xrightarrow{\text{env}_\Phi^\Omega Y \circ \varepsilon} & \text{Env}_\Phi^\Omega Y \\
 \varepsilon \searrow & & \swarrow \text{env}_\Phi^\Omega Y \\
 & Y & \\
 \varphi \searrow & \downarrow \varphi & \swarrow \varphi' \\
 & B &
 \end{array}$$

It must be understood as follows. On the one hand, since $\text{env}_{\Phi}^{\Omega} Y$ is an extension with respect to Φ , there exists a morphism φ' , such that the lower right triangle is commutative, and as a corollary, the perimeter is commutative as well. On the other hand, if φ' is a morphism such that the perimeter is commutative, i.e.

$$\varphi' \circ \text{env}_{\Phi}^{\Omega} Y \circ \varepsilon = \varphi \circ \varepsilon,$$

then, since ε is an epimorphism, we can cancel it:

$$\varphi' \circ \text{env}_{\Phi}^{\Omega} Y = \varphi,$$

So the lower right triangle must be commutative as well. This means that φ' must be unique (since by the definition of envelope, the dotted arrow in the lower right triangle must be unique).

We see that the perimeter has a unique dotted arrow φ' . This is true for any $\varphi \in \Phi$, and in addition $\text{env}_{\Phi}^{\Omega} Y \circ \varepsilon \in \Omega$. So we come to a conclusion that the morphism $\text{env}_{\Phi}^{\Omega} Y \circ \varepsilon$ is an extension of X in Ω with respect to the class of morphisms $\Phi \circ \varepsilon$. As a corollary, there exists a unique morphism v from $\text{Env}_{\Phi}^{\Omega} Y$ into the envelope $\text{Env}_{\Phi \circ \varepsilon}^{\Omega} X$ with respect to $\Phi \circ \varepsilon$ such that (1.12) is commutative. \square

- Let us say that in a category \mathbf{K} a class of morphisms Φ is generated on the inside by a class of morphisms Ψ , if

$$\Psi \subseteq \Phi \subseteq \text{Mor}(\mathbf{K}) \circ \Psi. \quad (1.13)$$

Theorem 1.1. Suppose that in a category \mathbf{K} a class of morphisms Φ is generated on the inside by a class of morphisms Ψ . Then for any class of epimorphisms Ω (it is not necessary that Ω contains all epimorphisms of \mathbf{K}) and for any object X the existence of envelope $\text{env}_{\Psi}^{\Omega} X$ is equivalent to the existence of envelope $\text{env}_{\Phi}^{\Omega} X$, and these envelopes coincide:

$$\text{env}_{\Psi}^{\Omega} X = \text{env}_{\Phi}^{\Omega} X. \quad (1.14)$$

Proof. 1. Suppose first that $\text{env}_{\Psi}^{\Omega} X$ exists. Since it is an extension with respect to Ψ , and at the same time an epimorphism, by 3° (a) we have that it is an extension with respect to Φ as well. If $\sigma : X \rightarrow X'$ is another extension with respect to Φ , then by 2° (a) it is an extension with respect to Ψ as well, so there exists a unique morphism $v : \text{Env}_{\Psi}^{\Omega} X \leftarrow X'$ such that the following diagram is commutative:

$$\begin{array}{ccc} & X & \\ \text{env}_{\Psi}^{\Omega} X \swarrow & & \searrow \sigma \\ \text{Env}_{\Psi}^{\Omega} X & \xleftarrow{\exists! v} & X' \end{array}$$

This means that $\text{env}_{\Psi}^{\Omega} X$ is an envelope with respect to Φ , and (1.14) holds.

2. On the contrary, suppose that $\text{env}_{\Phi}^{\Omega} X$ exists. It is an extension with respect to Φ , so by 2° (a) it must be an extension with respect to Ψ as well. If $\sigma : X \rightarrow X'$ is another extension in Ω with respect to Ψ , then, since $\sigma \in \text{Epi}$, by 3° (a), it must be an extension with respect to Φ , so there exists a unique morphism $v : X' \rightarrow \text{Env}_{\Phi}^{\Omega} X$ such that

$$\begin{array}{ccc} & X & \\ \sigma \swarrow & & \searrow \text{env}_{\Phi}^{\Omega} X \\ X' & \xrightarrow{\exists! v} & \text{Env}_{\Phi}^{\Omega} X \end{array}$$

This means that $\text{env}_{\Phi}^{\Omega} X$ is an envelope with respect to Ψ , and again we have (1.14). \square

- Let us say that a class of morphisms Φ in a category \mathbf{K} differs morphism on the outside, if for any two different parallel morphisms $\alpha \neq \beta : X \rightarrow Y$ there is a morphism $\varphi : Y \rightarrow M$ from the class Φ such that $\varphi \circ \alpha \neq \varphi \circ \beta$.

Theorem 1.2. If a class of morphisms Φ differs morphisms on the outside, then for any class of morphisms Ω

- (i) each extension in Ω with respect to Φ is a monomorphism,
- (ii) an envelope with respect to Φ in Ω exists if and only if there exists an envelope with respect to Φ in the class $\Omega \cap \text{Mono}$; in this case these envelopes coincide:

$$\text{env}_{\Phi}^{\Omega} = \text{env}_{\Phi}^{\Omega \cap \text{Mono}},$$

(iii) if the class Ω contains all monomorphisms,

$$\Omega \supseteq \text{Mono},$$

then the existence of the envelope with respect to Φ in Mono automatically implies the existence of envelope with respect to Φ in Ω , and the coincidence of these envelopes:

$$\text{env}_{\Phi}^{\Omega} = \text{env}_{\Phi}^{\text{Mono}}.$$

Proof. 1. Suppose that some extension $\sigma : X \rightarrow X'$ is not a monomorphism, i.e. there are two different parallel morphisms $\alpha \neq \beta : T \rightarrow X$ such that

$$\sigma \circ \alpha = \sigma \circ \beta. \quad (1.15)$$

Since the class Φ differs morphisms on the outside, there must exist a morphism $\varphi : X \rightarrow M$, $\varphi \in \Phi$, such that

$$\varphi \circ \alpha \neq \varphi \circ \beta. \quad (1.16)$$

Since $\sigma : X \rightarrow X'$ is an extension with respect to Φ , there is a continuation $\varphi' : X' \rightarrow M$ of the morphism $\varphi : X \rightarrow M$: $\varphi = \varphi' \circ \sigma$. Now we obtain

$$\varphi \circ \alpha = \varphi' \circ \sigma \circ \alpha = (1.15) = \varphi' \circ \sigma \circ \beta = \varphi \circ \beta,$$

and this contradicts to (1.16).

2. Suppose for an object X there exists an envelope $\text{env}_{\Phi}^{\Omega} X$ in Ω with respect to Φ . Then, as we have already proved, it must be an extension in the narrower class $\Omega \cap \text{Mono}$ with respect to Φ . Applying property 1° (c) on p.20, we obtain that $\text{env}_{\Phi}^{\Omega} X$ is an envelope in the narrower class $\Omega \cap \text{Mono}$, i.e. $\text{env}_{\Phi}^{\Omega} X = \text{env}_{\Phi}^{\Omega \cap \text{Mono}} X$.

Conversely, suppose there is an envelope $\text{env}_{\Phi}^{\Omega \cap \text{Mono}} X$ with respect to Φ in the class $\Omega \cap \text{Mono}$. By 1° (a) on p.20, it must be an extension with respect to Φ in the wider class Ω . Consider another extension $\sigma : X \rightarrow X'$ with respect to Φ in Ω . By the proposition (i) which we have already proved, σ is an extension with respect to Φ in $\Omega \cap \text{Mono}$. Hence, there is a unique morphism $v : X' \rightarrow \text{env}_{\Phi}^{\Omega \cap \text{Mono}} X$ into the envelope in the class $\Omega \cap \text{Mono}$, such that

$$\begin{array}{ccc} & X & \\ \sigma \swarrow & & \searrow \text{env}_{\Phi}^{\Omega \cap \text{Mono}} X \\ X' & \text{---} \text{ } \underset{v}{\text{---}} \text{ } \rightarrow & \text{env}_{\Phi}^{\Omega \cap \text{Mono}} X \end{array}$$

This proves that $\text{env}_{\Phi}^{\Omega \cap \text{Mono}} X$ is (not only an extension, but also) an envelope with respect to Φ in the class Ω .

3. The proposition (iii) immediately follows from (ii). \square

- Let us remind that a class of morphisms Φ in a category \mathbf{K} is called a *right ideal*, if

$$\Phi \circ \text{Mor}(\mathbf{K}) \subseteq \Phi$$

(i.e. for any $\varphi \in \Phi$ and for any morphism μ in \mathbf{K} the composition $\varphi \circ \mu$ belongs to Φ).

Theorem 1.3. *If a class of morphisms Φ differs morphism on the outside and is a right ideal in the category \mathbf{K} , then for any class of morphisms Ω*

- (i) *each extension in Ω with respect to Φ is a bimorphism,*
- (ii) *an envelope with respect to Φ in Ω exists if and only if there exists an envelope with respect to Φ in the class $\Omega \cap \text{Bim}$ of bimorphisms lying in Ω ; in this case these envelopes coincide:*

$$\text{env}_{\Phi}^{\Omega} = \text{env}_{\Phi}^{\Omega \cap \text{Bim}}.$$

(iii) if the class Ω contains all bimorphisms,

$$\Omega \supseteq \text{Bim},$$

then the existence of an envelope with respect to Φ in Bim automatically implies the existence of an envelope with respect to Φ in Ω and the coincidence of these envelopes:

$$\text{env}_{\Phi}^{\Omega} = \text{env}_{\Phi}^{\text{Bim}}.$$

Proof. 1. Let $\sigma : X \rightarrow X'$ be an extension in Ω with respect to Φ . It is already proved in Theorem 1.2 (i) that σ must be a monomorphism. Suppose that it is not an epimorphism. This means that there are two different parallel morphisms $\alpha \neq \beta : X' \rightarrow T$ such that

$$\alpha \circ \sigma = \beta \circ \sigma. \quad (1.17)$$

Since Φ differs morphisms on the outside, there must exist a morphism $\varphi : T \rightarrow M$, $\varphi \in \Phi$, such that

$$\varphi \circ \alpha \neq \varphi \circ \beta.$$

In addition, by (1.17),

$$\varphi \circ \alpha \circ \sigma = \varphi \circ \beta \circ \sigma.$$

If we now suppose that Φ is a right ideal in K , then the morphism $\varphi \circ \alpha \circ \sigma = \varphi \circ \beta \circ \sigma$ lies in Φ . So we can interpret this picture as follows: the test (i.e. lying in Φ) morphism $\varphi \circ \alpha \circ \sigma = \varphi \circ \beta \circ \sigma : X \rightarrow M$ has two different continuations $\varphi \circ \alpha \neq \varphi \circ \beta : X' \rightarrow M$ along $\sigma : X \rightarrow X'$. This means that σ cannot be an extension with respect to Φ .

2. The second and the third parts of the theorem are proved by the same reasoning as in Theorem 1.2. \square

Connection with projective limits. The similarity between the notions of an envelope and a projective limit is formalized in the following

Lemma 1.1. *The projective limit $\rho = \varprojlim \rho^i : X \rightarrow \varprojlim X^i$ of any projective cone $\{\rho^i : X \rightarrow X^i; i \in I\}$ from a given object X into a covariant (or contravariant) system $\{X^i, \iota_i^j\}$ is an envelope of the object X in an arbitrary class Ω containing ρ with respect to the system of morphisms $\{\rho^i; i \in I\}$:*

$$\rho = \varprojlim \rho^i \in \Omega \implies \text{Env}_{\{\rho^i; i \in I\}}^\Omega X = \varprojlim X^i \quad (1.18)$$

In particular, this is always true for $\Omega = \text{Mor}(K)$:

$$\text{Env}_{\{\rho^i; i \in I\}} X = \varprojlim X^i \quad (1.19)$$

Proof. 1. First, the morphism ρ is an extension of X with respect to the system $\{\rho^i\}$, since the definition of projective limit guarantees that for any ρ^j there exists a unique continuation π^j on $\varprojlim X^i$:

$$\begin{array}{ccc} X & \xrightarrow{\rho} & \varprojlim X^i \\ & \searrow \rho^j & \swarrow \pi^j \\ & X^j & \end{array} \quad (1.20)$$

2. Suppose then that $\sigma : X \rightarrow X'$ is another extension. Then for any morphism $\rho^j : X \rightarrow X^j$ there is a unique morphism $v^j : X' \rightarrow X^j$, such that

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X' \\ & \searrow \rho^j & \swarrow v^j \\ & X^j & \end{array} \quad (1.21)$$

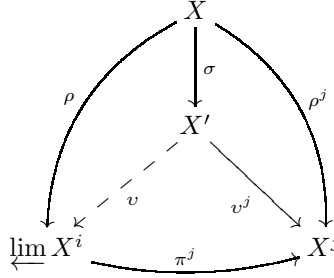
For each indices $i \leq j$ in the following diagram

$$\begin{array}{ccccc} & & X & & \\ & \swarrow \rho^i & \downarrow \sigma & \searrow \rho^j & \\ & & X' & & \\ & \swarrow v^i & & \searrow v^j & \\ X^i & & & & X^j \\ & \swarrow \iota_i^j & & \searrow & \end{array}$$

the following elements will be commutative: two upper little triangles (each one has one dotted arrow) and the perimeter (without dotted arrows). This together with the uniqueness of v^j in the upper right little triangle implies that the lower little triangle (with two dotted arrows) is commutative as well:

$$\begin{cases} (\iota_i^j \circ v^i) \circ \sigma = \iota_i^j \circ (v^i \circ \sigma) = \iota_i^j \circ \rho^i = \rho^j \\ v^j \circ \sigma = \rho^j \end{cases} \implies \iota_i^j \circ v^i = v^j.$$

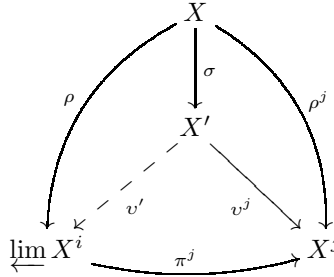
The commutativity of the triangle with two dotted arrows means in its turn that X' with the system of morphisms v^i is a projective cone of the covariant system $\{X^i; \iota_i^j\}$. So there must exist a uniquely defined morphism v such that for any j in the diagram



the little lower triangle is commutative. On the other hand, the right little triangle here is also commutative, since this is diagram (1.21) turned around, and the perimeter is commutative, since this is diagram (1.20) turned around. Together with the uniqueness of the morphism ρ in the system of all those perimeters with different j this implies that the left little triangle is also commutative:

$$\left(\forall j \quad \begin{cases} \pi^j \circ v \circ \sigma = v^j \circ \sigma = \rho^j \\ \pi^j \circ \rho = \rho^j \end{cases} \right) \implies v \circ \sigma = \rho$$

We understood that there is a morphism v such that diagram (1.6) is commutative (with $E = \varprojlim X^i$). It remains to verify that such a morphism is unique. Let v' be another morphism with the same property: $\rho = v' \circ \sigma$. Consider the following diagram:



Here (apart from the upper left triangle) the upper right triangle will be commutative (since this is the turned around diagram (1.21)) and the perimeter as well (since this is diagram (1.20) turned around). Together with the uniqueness of the arrow v^j in the upper right triangle this implies that the lower little triangle is also commutative:

$$\begin{cases} \pi^j \circ v' \circ \sigma = \pi^j \circ \rho = \rho^j \\ v^j \circ \sigma = \rho^j \end{cases} \implies \pi^j \circ v' = v^j.$$

This is true for each index j , so the morphism v' must coincide with the morphism v which we constructed before: $v' = v$. \square

Envelope in a class of objects with respect to a class of objects. A special case of the construction is a situation, where Ω and/or Φ are classes of all morphisms into the objects from some given subclasses in $\mathbf{Ob}(\mathbf{K})$. The accurate formulation for the case, when both classes Ω and Φ are defined in such a way is the following. Suppose we have a category \mathbf{K} and two subclasses \mathbf{L} and \mathbf{M} in the class $\mathbf{Ob}(\mathbf{K})$ of objects in \mathbf{K} .

- A morphism $\sigma : X \rightarrow X'$ is called an *extension of the object $X \in \mathbf{K}$ in the class \mathbf{L} with respect to the class \mathbf{M}* , if $X' \in \mathbf{L}$ and for any object $B \in \mathbf{M}$ and any morphism $\varphi : X \rightarrow B$ there exists a unique morphism

$\varphi' : X' \rightarrow B$ such that the following diagram is commutative:

$$\begin{array}{ccc} & X & \\ \sigma \swarrow & & \searrow \forall \varphi \\ X' & \xrightarrow{\quad \exists! \varphi' \quad} & B \\ \cap & & \cap \\ L & & M \end{array}$$

- An extension $\rho : X \rightarrow E$ of an object $X \in K$ in the class L with respect to the class M is called an *envelope of the object $X \in K$ in the class L with respect to the class M* , and we denote this by formula

$$\rho = \text{env}_M^L X, \quad (1.22)$$

if for any other extension $\sigma : X \rightarrow X'$ (of the object X in the class L with respect to the class M) there exists a unique morphism $v : X' \rightarrow E$ such that the following diagram is commutative:

$$\begin{array}{ccc} & X & \\ \forall \sigma \swarrow & & \searrow \rho \\ X' & \xrightarrow{\quad \exists! v \quad} & E \\ \cap & & \cap \\ L & & L \end{array} \quad (1.23)$$

The object E is also called an *envelope* of the object X (in the class of objects L with respect to the class of objects M), and we will use the following notation for it:

$$E = \text{Env}_M^L X. \quad (1.24)$$

The following two extreme situations in the choice of the subclass L can occur:

- if $L = \text{Ob}(K)$, then we will speak about an *envelope of the object $X \in K$ in the category K with respect to the class of objects M* , and the notations will be the following:

$$\text{env}_M X := \text{env}_M^K X, \quad \text{Env}_M X := \text{Env}_M^K X. \quad (1.25)$$

- if $L = M$, then the notions of the extension and the envelope coincide: *each extension of the object X in the class L with respect to the same class L is an envelope of X in the class L with respect to L* (indeed, if $\rho : X \rightarrow E$ and $\sigma : X \rightarrow X'$ are two extensions in L with respect to L , then in diagram (1.23) the morphism v exists and is unique just because σ is an extension); for simplicity, in case of $L = M$ we speak about the *envelope of X in the class L* , and our notations are simplified as follows:

$$\text{env}^L X := \text{env}_L^L X, \quad \text{Env}^L X := \text{Env}_L^L X. \quad (1.26)$$

- Let us say that a class of objects M in a category K *differs morphisms on the outside*, if the class of morphisms with ranges in M possesses this property (in the sense of definition on page 22), i.e. for any two different parallel morphisms $\alpha \neq \beta : X \rightarrow Y$ there is a morphism $\varphi : Y \rightarrow M$ such that $\varphi \circ \alpha \neq \varphi \circ \beta$.

From Theorem 1.3 we have

Theorem 1.4. *If a class of objects M differs morphisms on the outside, then for any class of objects L*

- (i) *each envelope in L with respect to M is a bimorphism,*
- (ii) *an envelope in L with respect to M exists if and only if there exists an anvelope in the class of bimorphisms with the values in L with respect to M ; in this case these envelopes coincide:*

$$\text{env}_M^L = \text{env}_M^{\text{Bim}(K,L)}.$$

Examples of envelopes.

Example 1.1. Universal enveloping algebra. Let $\mathbf{K} = \mathbf{LieAlg}$ be the category of Lie algebras (say, over the field \mathbb{C}), $\mathbf{T} = \mathbf{Alg}$ the category of associative algebras (again, over \mathbb{C}) with the identity, and $F : \mathbf{Alg} \rightarrow \mathbf{LieAlg}$ the functor that each associative algebra A represents as the Lie algebra with the Lie brackets

$$[x, y] = x \cdot y - y \cdot x.$$

Then the envelope of a Lie algebra \mathfrak{g} over the category \mathbf{Alg} in the class $\mathbf{Mor}(\mathbf{LieAlg}, F(\mathbf{Alg}))$ of all morphisms from \mathbf{LieAlg} into $F(\mathbf{Alg})$ with respect to the same class $\mathbf{Mor}(\mathbf{LieAlg}, F(\mathbf{Alg}))$ is exactly the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra \mathfrak{g} (cf.[9]):

$$U(\mathfrak{g}) = \mathbf{Env}^{\mathbf{Mor}(\mathbf{LieAlg}, F(\mathbf{Alg}))} \mathfrak{g}.$$

Example 1.2. Stone-Čech compactification. In the category \mathbf{Tikh} of Tikhonov spaces the Stone-Čech compactification $\beta : X \rightarrow \beta X$ is an envelope of the space X in the class \mathbf{Com} of compact spaces with respect to the same class \mathbf{Com} :

$$\beta X = \mathbf{Env}^{\mathbf{Com}} X.$$

Proof. Here one uses the theorem [39, Theorem 3.6.1], which states that any continuous map $f : X \rightarrow K$ into an arbitrary compact space K can be extended to a continuous map $F : \beta X \rightarrow K$. Since $\beta(X)$ is dense in βX , this extension F is unique, and this means that $\beta : X \rightarrow \beta X$ is an extension in the class \mathbf{Com} with respect to \mathbf{Com} . By the Remark on p.26, in the case $\mathbf{L} = \mathbf{M}$ each extension is an envelope, so β is an envelope. \square

Example 1.3. Completion X^∇ of a locally convex space X is an envelope of X in the category \mathbf{LCS} of all locally convex spaces with respect to the class \mathbf{Ban} of Banach spaces:

$$X^\nabla = \mathbf{Env}_{\mathbf{Ban}}^{\mathbf{LCS}} X.$$

Proof. Let us denote the natural embedding of X into its completion by $\nabla_X : X \rightarrow X^\nabla$ (we use the notations of [2]).

First, each linear continuous map $f : X \rightarrow B$ into an arbitrary Banach space B is uniquely extended to a linear continuous map $F : X^\nabla \rightarrow B$ on the completion X^∇ of X (one can refer here, for instance, to the general theorem for all uniform spaces [39, Theorem 8.3.10]). Hence, the completion $\nabla_X : X \rightarrow X^\nabla$ is an extension of the space X in the category \mathbf{LCS} of locally convex spaces with respect to the subclass \mathbf{Ban} of Banach spaces.

Note then, that the class \mathbf{Ban} of Banach spaces differs morphisms on the outside in the category \mathbf{LCS} . By Theorem 1.4 this means that any extension $\sigma : X \rightarrow X'$ with respect to \mathbf{Ban} is a bimorphism in \mathbf{LCS} , i.e. is an injective map which image $\sigma(X)$ is dense in X' . Let us show that in addition it is an open map: for any neighborhood of zero $U \subseteq X$ there is a neighborhood of zero $V \subseteq X'$ such that

$$\sigma(U) \supseteq V \cap \sigma(X) \tag{1.27}$$

We can think that U is a closed convex neighborhood of zero in X . Then the set $\mathbf{Ker} U = \bigcap_{\varepsilon > 0} \varepsilon \cdot U$ is a closed subspace in X . Consider the quotient space $X/\mathbf{Ker} U$ and endow it with the topology of normed space with the unit ball $U + \mathbf{Ker} U$. Then $(X/\mathbf{Ker} U)^\nabla$ will be a Banach space, and we denote it by X/U . The natural map (the composition of the quotient map $X \rightarrow X/\mathbf{Ker} U$ and the completion $X/\mathbf{Ker} U \rightarrow (X/\mathbf{Ker} U)^\nabla$) will be denoted by $\pi_U : X \rightarrow X/U$. Since $\sigma : X \rightarrow X'$ is an extension with respect to \mathbf{Ban} , the map $\pi_U : X \rightarrow X/U$ is extended to some linear continuous map $(\pi_U)' : X' \rightarrow X/U$.

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X' \\ \pi_U \searrow & & \swarrow (\pi_U)' \\ & X/U & \end{array}$$

If we denote by W the unit ball in X/U , i.e the closure of the set $U + \mathbf{Ker} U$ in the space $(X/\mathbf{Ker} U)^\nabla = X/U$, then for the neighborhood of zero $V = ((\pi_U)')^{-1}(W)$ we obtain the following chain, which proves (1.27):

$$\begin{aligned} y \in V \cap \sigma(X) &\Rightarrow \exists x \in X : y = \sigma(x) \ \& \ y \in V \Rightarrow \\ \Rightarrow \exists x \in X : y = \sigma(x) \ \& \ (\pi_U)'(y) = (\pi_U)'(\sigma(x)) = \underbrace{\pi_U(x)}_{\substack{\updownarrow \\ x \in U}} \in W &\Rightarrow \exists x \in U : y = \sigma(x) \Rightarrow y \in \sigma(U). \end{aligned}$$

Thus, $\sigma : X \rightarrow X'$ is an open and injective linear continuous map, and $\sigma(X)$ is dense in X' . This means that X' can be perceived as a subspace in the completion X^∇ of the space X with the topology induced from X^∇ . I.e. there is a unique linear continuous map $v : X' \rightarrow X^\nabla$ such that

$$\begin{array}{ccc} & X & \\ \sigma \swarrow & & \searrow \nabla_X \\ X' & \xrightarrow{\quad v \quad} & X^\nabla \end{array}$$

We conclude that $\nabla_X : X \rightarrow X^\nabla$ is an envelope of X in LCS with respect to **Ban**. \square

(b) Imprint

Imprint of a class of morphisms by means of a class of morphisms. Suppose we have:

- a category \mathbf{K} , called *enveloping category*,
- a category \mathbf{T} , called *repelling category*,
- a covariant functor $F : \mathbf{T} \rightarrow \mathbf{K}$,
- two classes Ω and Φ of morphisms in \mathbf{K} , which have domains in objects of the class $F(\mathbf{T})$, and Ω is called a *class of realizing morphisms*, and Φ a *class of test morphisms*.

Then

- For given objects $X \in \text{Ob}(\mathbf{K})$ and $X' \in \text{Ob}(\mathbf{T})$ a morphism $\sigma : F(X') \rightarrow X$ is called a *domain of influence of a class of morphisms Ω in the object $X \in \mathbf{K}$ over the category \mathbf{T} by means of the class of morphisms Φ* , if $\sigma \in \Omega$, and for any morphism $\varphi : B \rightarrow X$, $\varphi \in \Phi$, there is a unique morphism $\varphi' : B \rightarrow X'$ such that the following diagram is commutative:

$$\begin{array}{ccc} & X & \\ \Phi \ni \varphi \swarrow & & \nwarrow \sigma \in \Omega \\ F(B) & \xrightarrow{\quad F(\varphi') \quad} & F(X') \end{array} \quad (1.28)$$

- A domain of influence $\rho : F(E) \rightarrow X$ of the class of morphisms Ω in an object $X \in \mathbf{K}$ over the category \mathbf{T} by means of the class of morphisms Φ is called an *imprint of a class of morphisms Ω in the object $X \in \mathbf{K}$ over the category \mathbf{T} by means of the class of morphisms Φ* , and we denote this by formula

$$\rho = \text{imp}_\Phi^\Omega X, \quad (1.29)$$

if for any other domain of influence $\sigma : F(X') \rightarrow X$ (of the class of morphisms Ω in the object $X \in \mathbf{K}$ over the category \mathbf{T} by means of the class of morphisms Φ) there is a unique morphism $v : E \rightarrow X'$ in \mathbf{T} such that the following diagram is commutative:

$$\begin{array}{ccc} & X & \\ \rho \swarrow & & \nwarrow \sigma \\ F(E) & \xrightarrow{\quad F(v) \quad} & F(X') \end{array} \quad (1.30)$$

The very object E is also called an *imprint* of a class of morphisms Ω in the object $X \in \mathbf{K}$ over the category \mathbf{T} by means of the class of morphisms Φ , and we denote this by the following formula:

$$E = \text{Imp}_\Phi^\Omega X. \quad (1.31)$$

Further we are almost exclusively interested in the case when $\mathbf{T} = \mathbf{K}$, and $F : \mathbf{K} \rightarrow \mathbf{K}$ is the identity functor. In doing this we will omit in terminology the mentioning of the category \mathbf{T} , so we'll be speaking about *domain of*

influence and about *imprint* of the class of morphisms Ω by means of the class of morphisms Φ . The diagrams (1.28) and (1.30) will turn respectively into the diagrams

$$\begin{array}{ccc} & X & \\ \nearrow \forall \varphi \in \Phi & & \nwarrow \sigma \in \Omega \\ B & \dashrightarrow_{\exists! \varphi'} & X' \end{array} \quad (1.32)$$

and

$$\begin{array}{ccc} & X & \\ \nearrow \rho & & \nwarrow \forall \sigma \\ E & \dashrightarrow_{\exists! v} & X' \end{array} \quad (1.33)$$

Remark 1.5. Like in the case of envelope, the imprint $\text{Imp}_{\Phi}^{\Omega} X$ (if exists) is defined up to an isomorphism. If we want to have a map $X \mapsto \text{Imp}_{\Phi}^{\Omega} X$, then (providing that the imprint always exists) the sufficient condition for existence of this map is the skeletal smallness of the category \mathbf{K} . In the important special case when Ω is the class $\text{Mono}(\mathbf{K})$ of all monomorphisms in the category \mathbf{K} , the sufficient condition for existence of the map $X \mapsto \text{Imp}_{\Phi}^{\Omega} X$ (again, providing that the envelope always exists) is that the category \mathbf{K} is well-powered (since in this case the object $\text{Imp}_{\Phi}^{\text{Epi}} X$ can be chosen by the axiom of choice as a concrete subobject of X).

Remark 1.6. If $\Omega = \emptyset$, then, of course, neither domains of influence, no imprints of the class Ω exist in the objects of the category \mathbf{K} . So this construction is interesting only if Ω is a non-empty class. The following two situations will be of special interest:

- $\Omega = \text{Mono}(\mathbf{K})$ (i.e. Ω coincides with the class of all monomorphisms of the category \mathbf{K}), then we will use the following notations:

$$\text{imp}_{\Phi}^{\text{Mono}} X := \text{imp}_{\Phi}^{\text{Mono}(\mathbf{K})} X, \quad \text{Imp}_{\Phi}^{\text{Mono}} X := \text{Imp}_{\Phi}^{\text{Mono}(\mathbf{K})} X. \quad (1.34)$$

- $\Omega = \text{Mor}(\mathbf{K})$ (i.e. Ω coincides with the class of all morphisms of the category \mathbf{K}), in this case it is convenient to omit any mentioning about Ω in the formulations and notations, so we will be speaking about *imprints of the category \mathbf{K} in the object $X \in \mathbf{K}$ by means of the class of morphisms Φ* , and the notations will be simplified as follows:

$$\text{imp}_{\Phi} X := \text{imp}_{\Phi}^{\text{Mor}(\mathbf{K})} X, \quad \text{Imp}_{\Phi} X := \text{Imp}_{\Phi}^{\text{Mor}(\mathbf{K})} X. \quad (1.35)$$

Remark 1.7. Another degenerate, but this time an informative case is when $\Phi = \emptyset$. What is essential for a given object X , Φ does not contain morphisms coming to X :

$$\Phi^X = \{\varphi \in \Phi : \text{Ran}(\varphi) = X\} = \emptyset.$$

Then, obviously, any morphism $\sigma \in \Omega$ coming to X , $\sigma : X \leftarrow X'$, is a domain of influence (of the class of morphisms Ω) in X (by means of the class of morphisms \emptyset). If in addition $\Omega = \text{Mono}$, then the imprint will be the initial object of the category $\text{Mono}(X)$ (if it exists). This can be depicted by the formula

$$\text{Imp}_{\emptyset}^{\Omega} X = \min \text{Mono}(X).$$

On the other hand, if \mathbf{K} is a category with zero 0 , and Ω contains all the morphisms going from 0 , then the imprint of Ω in each object by means of the empty class of morphisms is 0 :

$$\text{Imp}_{\emptyset}^{\Omega} X = 0.$$

Remark 1.8. Another extreme situation is when $\Phi = \text{Mor}(\mathbf{K})$. For a given object X the essential thing here is that Φ contains the local identity for X :

$$1_X \in \Phi.$$

Then for any domain of influence σ the diagram

$$\begin{array}{ccc} X & \xleftarrow{\sigma} & X' \\ & \nwarrow 1_X & \nearrow \\ & B & \end{array}$$

implies that σ must be a retraction (moreover, the dotted arrow here must be unique). In the special case if $\Omega = \text{Mono}$ this is possible only if σ is an isomorphism. As a corollary, the imprint of Ω in X coincides here with X (up to isomorphism):

$$\text{Imp}_{\text{Mor}(\mathbb{K})}^{\text{Mono}} X = X.$$

Properties of imprints:

1°. Suppose $\Sigma \subseteq \Omega$, then for any object X and for any class of morphisms Φ

- (a) each domain of influence $\sigma : X \leftarrow X'$ of the Σ by means of Φ is a domain of influence of Ω by means of Φ ,
- (b) if there are imprints $\text{imp}_{\Phi}^{\Sigma} X$ and $\text{imp}_{\Phi}^{\Omega} X$, then there is a unique morphism $\rho : \text{Imp}_{\Phi}^{\Sigma} X \leftarrow \text{Imp}_{\Phi}^{\Omega} X$ such that the following diagram is commutative:

$$\begin{array}{ccc} & X & \\ \text{imp}_{\Phi}^{\Sigma} X \nearrow & & \nwarrow \text{imp}_{\Phi}^{\Omega} X \\ \text{Imp}_{\Phi}^{\Sigma} X & \xleftarrow{\rho} & \text{Imp}_{\Phi}^{\Omega} X \end{array} \quad (1.36)$$

- (c) if there is an imprint $\text{imp}_{\Phi}^{\Omega} X$ (of a wider class), and it lies in the (narrower) class Σ ,

$$\text{imp}_{\Phi}^{\Omega} X \in \Sigma,$$

then it is an imprint $\text{imp}_{\Phi}^{\Sigma} X$ (of a narrower class):

$$\text{imp}_{\Phi}^{\Omega} X = \text{imp}_{\Phi}^{\Sigma} X.$$

2°. Suppose $\Psi \subseteq \Phi$, then for any object X and for any class of morphisms Ω

- (a) each domain of influence $\sigma : X \leftarrow X'$ of Ω in X by means of Φ is a domain of influence of Ω in X by means of Ψ ,
- (b) if there are imprints $\text{Imp}_{\Psi}^{\Omega} X$ and $\text{Imp}_{\Phi}^{\Omega} X$, then there is a unique morphism $\alpha : \text{Imp}_{\Psi}^{\Omega} X \rightarrow \text{Imp}_{\Phi}^{\Omega} X$ such that the following diagram is commutative:

$$\begin{array}{ccc} & X & \\ \text{imp}_{\Psi}^{\Omega} X \nearrow & & \nwarrow \text{imp}_{\Phi}^{\Omega} X \\ \text{Imp}_{\Psi}^{\Omega} X & \xrightarrow{\alpha} & \text{Imp}_{\Phi}^{\Omega} X \end{array} \quad (1.37)$$

3°. Suppose $\Phi \subseteq \Psi \circ \text{Mor}(\mathbb{K})$ (i.e. each morphism $\varphi \in \Phi$ can be represented in the form $\varphi = \psi \circ \chi$, where $\psi \in \Psi$), then for any object X and for any class of morphisms Ω

- (a) if a domain of influence $\sigma : X \leftarrow X'$ of Ω by means of Ψ is at the same time a monomorphism in \mathbb{K} , then it is a domain of influence of Ω by means of Φ ,
- (b) if there are imprints $\text{imp}_{\Psi}^{\Omega} X$ and $\text{imp}_{\Phi}^{\Omega} X$, and $\text{imp}_{\Psi}^{\Omega} X$ is at the same time a monomorphism in \mathbb{K} , then there is a unique morphism $\beta : \text{Imp}_{\Psi}^{\Omega} X \leftarrow \text{Imp}_{\Phi}^{\Omega} X$ such that the following diagram is commutative:

$$\begin{array}{ccc} & X & \\ \text{imp}_{\Psi}^{\Omega} X \nearrow & & \nwarrow \text{imp}_{\Phi}^{\Omega} X \\ \text{Imp}_{\Psi}^{\Omega} X & \xleftarrow{\beta} & \text{Imp}_{\Phi}^{\Omega} X \end{array} \quad (1.38)$$

4°. Let the classes of morphisms Ω , Φ and a monomorphism $\mu : X \leftarrow Y$ in \mathbb{K} satisfy the following conditions:

- (a) there is an imprint $\text{Imp}_{\mu \circ \Phi}^{\Omega} X$ by means if the class of morphisms $\mu \circ \Phi = \{\mu \circ \varphi; \varphi \in \Phi\}$,
- (b) there is an imprint $\text{Imp}_{\Phi}^{\Omega} Y$,
- (c) the composition $\mu \circ \text{imp}_{\Phi}^{\Omega} Y$ belongs to Ω .

Then there is a unique morphism $v : \text{Imp}_{\mu \circ \Phi}^\Omega X \rightarrow \text{Imp}_\Phi^\Omega Y$ such that the following diagram is commutative:

$$\begin{array}{ccc}
 X & \xleftarrow{\mu} & Y \\
 \uparrow \text{imp}_{\mu \circ \Phi}^\Omega X & \swarrow \mu \circ \text{imp}_\Phi^\Omega Y & \uparrow \text{imp}_\Phi^\Omega Y \\
 \text{Imp}_{\mu \circ \Phi}^\Omega X & \xrightarrow{\quad v \quad} & \text{Imp}_\Phi^\Omega Y
 \end{array} \tag{1.39}$$

- Let us say that in a category K a class of morphisms Φ is generated on the outside by a class of morphisms Ψ , if

$$\Psi \subseteq \Phi \subseteq \Psi \circ \text{Mor}(K).$$

The following fact is dual to Theorem 1.1 and is proved by analogy:

Theorem 1.5. Suppose in a category K a class of morphisms Φ is generated on the outside by a class of morphisms Ψ . Then for any class of monomorphisms Ω (it is not necessary that Ω contains all monomorphisms of the category K) and for any object X the existence of imprint $\text{imp}_\Psi^\Omega X$ is equivalent to the existence of the imprint $\text{imp}_\Phi^\Omega X$, and these imprints coincide:

$$\text{imp}_\Psi^\Omega X = \text{imp}_\Phi^\Omega X. \tag{1.40}$$

- Let us say that a class of morphisms Φ in a category K differs morphisms on the inside, if for any two different parallel morphisms $\alpha \neq \beta : X \rightarrow Y$ there is a morphism $\varphi : M \rightarrow X$ from the class Φ such that $\alpha \circ \varphi \neq \beta \circ \varphi$.

The following result is dual to Theorem 1.2:

Theorem 1.6. If the class of morphisms Φ differs morphisms on the inside, then for any class of morphisms Ω

- (i) each domain of influence of Ω by means of Φ is an epimorphism,
- (ii) the imprint of Ω by means of Φ exists if and only if there exists an imprint of the class $\Omega \cap \text{Mono}$ by means of Φ ; in that case these imprints coincide:

$$\text{imp}_\Phi^\Omega = \text{imp}_\Phi^{\Omega \cap \text{Epi}},$$

- (iii) if the class Ω contains all epimorphisms,

$$\Omega \supseteq \text{Epi},$$

then the existence of an imprint of Epi by means of Φ automatically implies the existence of an imprint of Ω by means of Φ , and the coincidence of these imprints:

$$\text{imp}_\Phi^\Omega = \text{imp}_\Phi^{\text{Epi}}.$$

- Let us remind that a class of morphisms Φ in a category K is called a *left ideal*, if

$$\text{Mor}(K) \circ \Phi \subseteq \Phi$$

(i.e. for any $\varphi \in \Phi$ and for any morphism μ in K the composition $\mu \circ \varphi$ belongs to Φ).

The following proposition is dual to Theorem 1.3

Theorem 1.7. If a class of morphisms Φ differs morphisms on the inside and is a left ideal in the category K , then for any class of morphisms Ω

- (i) each domain of influence of Ω by means of Φ is a bimorphism,
- (ii) an imprint of Ω by means of Φ exists if and only if there exists an imprint of the class $\Omega \cap \text{Bim}$ of all bimorphisms lying in Ω by means of Φ ; in that case these imprints coincide:

$$\text{imp}_\Phi^\Omega = \text{imp}_\Phi^{\Omega \cap \text{Bim}}.$$

- (iii) if Ω contains all bimorphisms,

$$\Omega \supseteq \text{Bim},$$

then the existence of an imprint of Bim by means of Φ implies the existence of an imprint of Ω by means of Φ , and the coincidence of these imprints:

$$\text{imp}_\Phi^\Omega = \text{imp}_\Phi^{\text{Bim}}.$$

Connection with injective limit. The following lemma explains the similarity between the notions of imprint and that of injective limit, and is dual to Lemma 1.1.

Lemma 1.2. *The injective limit $\rho = \varinjlim \rho^i : X \leftarrow \varinjlim X^i$ of any injective cone $\{\rho^i : X \leftarrow X^i; i \in I\}$ into a given object X from a covariant (or contravariant) system $\{X^i, \iota_i^j\}$ is an imprint in the object X of an arbitrary class of object Ω containing ρ by means of the system of morphisms $\{\rho^i; i \in I\}$:*

$$\rho = \varinjlim \rho^i \in \Omega \implies \text{Imp}_{\{\rho^i; i \in I\}}^\Omega X = \varinjlim X^i \quad (1.41)$$

In particular, this is true for $\Omega = \text{Mor}(\mathbf{K})$:

$$\text{Imp}_{\{\rho^i; i \in I\}} X = \varinjlim X^i \quad (1.42)$$

Imprint of a class of objects by means of a class of objects. A special case is the situation when Ω and/or Φ are classes of all morphisms from some given subclass of objects in $\text{Ob}(\mathbf{K})$. An exact formulation for the case when both classes Ω and Φ are defined in this way is the following: suppose we have a category \mathbf{K} and two subclasses \mathbf{L} and \mathbf{M} in the class $\text{Ob}(\mathbf{K})$ of objects of \mathbf{K} .

- A morphism $\sigma : X' \rightarrow X$ is called a *domain of influence of the class of objects \mathbf{L} in the object $X \in \mathbf{K}$ by means of the class of objects \mathbf{M}* , if for any object $B \in \mathbf{M}$ and for any morphism $\varphi : B \rightarrow X$ there is a unique morphism $\varphi' : B \rightarrow X'$ such that the following diagram is commutative:

$$\begin{array}{ccc} & X & \\ \nearrow \forall \varphi & & \nwarrow \sigma \\ B & \xrightarrow{\quad \exists! \varphi' \quad} & X' \\ \cap & & \cap \\ \mathbf{M} & & \mathbf{L} \end{array}$$

- A domain of influence $\rho : E \rightarrow X$ of the class of objects \mathbf{L} in the object $X \in \mathbf{K}$ by means of the class of objects \mathbf{M} is called an *imprint of the class of objects \mathbf{L} in the object $X \in \mathbf{K}$ by means of the class of objects \mathbf{M}* , and we write in this case

$$\rho = \text{imp}_{\mathbf{M}}^{\mathbf{L}} X, \quad (1.43)$$

if for any other domain of influence $\sigma : X' \rightarrow X$ (of the class of objects \mathbf{L} in the object $X \in \mathbf{K}$ by means of the class of objects \mathbf{M}) there is a unique morphism $v : E \rightarrow X'$ such that the following diagram is commutative:

$$\begin{array}{ccc} & X & \\ \nearrow \rho & & \nwarrow \forall \sigma \\ E & \xrightarrow{\quad \exists! v \quad} & X' \\ \cap & & \cap \\ \mathbf{L} & & \mathbf{L} \end{array} \quad (1.44)$$

The very object E is also called an *imprint* of the class of objects \mathbf{L} in the object $X \in \mathbf{K}$ by means of the class of objects \mathbf{M} , and we use the following notation for it:

$$E = \text{Imp}_{\mathbf{M}}^{\mathbf{L}} X. \quad (1.45)$$

The following two extreme situations in the choice of \mathbf{L} can occur:

- if $\mathbf{L} = \text{Ob}(\mathbf{K})$, then we speak about an *imprint of the category \mathbf{K} in the object $X \in \mathbf{K}$ by means of the class of objects \mathbf{M}* , and the notations will be the following:

$$\text{imp}_{\mathbf{M}} X := \text{imp}_{\mathbf{M}}^{\mathbf{K}} X, \quad \text{Imp}_{\mathbf{M}} X := \text{Imp}_{\mathbf{M}}^{\mathbf{K}} X, \quad (1.46)$$

- if $\mathbf{L} = \mathbf{M}$, then the notions of the domain of influence and the imprint coincide: *each domain of influence of a class \mathbf{L} in an object $X \in \mathbf{K}$ by means of the very same class \mathbf{L} is an imprint of \mathbf{L} in X by means of \mathbf{L}* (since if $\rho : E \rightarrow X$ and $\sigma : X' \rightarrow X$ are two domains of influence of \mathbf{L} in X by means of \mathbf{L} , then in diagram (1.44) the morphism v exists and is unique just because σ is a domain of influence); for simplicity, in this case we will be speaking about *imprint of the class \mathbf{L} in the object X* , and the notations will be simplified as follows:

$$\text{imp}_{\mathbf{L}}^{\mathbf{L}} X =: \text{imp}^{\mathbf{L}} X, \quad \text{Imp}_{\mathbf{L}}^{\mathbf{L}} X =: \text{Imp}^{\mathbf{L}} X \quad (1.47)$$

- Let us say that a class of objects \mathbf{M} in the category \mathbf{K} *differs morphisms on the inside*, if the class of all morphisms going from objects of \mathbf{M} has this property (in the sense of definition on page 31), i.e. for any two different parallel morphisms $\alpha \neq \beta : X \rightarrow Y$ there is a morphism $\varphi : M \rightarrow X$ such that $\alpha \circ \varphi \neq \beta \circ \varphi$.

Theorem 1.7 implies

Theorem 1.8. *If a class of objects \mathbf{M} differs morphisms on the inside, then for any class of objects \mathbf{L}*

- (i) *each domain of convergence of the class \mathbf{L} by means of the class \mathbf{M} is a bimorphism,*
- (ii) *an imprint of the class \mathbf{L} by means of the class \mathbf{M} exists if and only if there exists an imprint of the class of bimorphisms going from \mathbf{L} by means of the class \mathbf{M} ; in that case these imprints coincide:*

$$\text{imp}_{\mathbf{M}}^{\mathbf{L}} = \text{imp}_{\mathbf{M}}^{\text{Bim}(\mathbf{L}, \mathbf{K})}.$$

Examples of imprints.

Example 1.4. Simply connected covering used in the theory of Lie groups is from the categorial point of view an imprint of the class of pointed coverings by means of empty class of morphisms in the category of connected locally connected and semilocally simply connected topological spaces (see definitions in [28]).

Example 1.5. Bornologification (see definition in [19]) X_{born} of a locally convex space X is an imprint in X of the category LCS of locally convex spaces by means of the subcategory Norm of normed spaces:

$$X_{\text{born}} = \text{Imp}_{\text{Norm}}^{\text{LCS}} X$$

Proof. This follows from the characterization of bornologification as the strongest locally convex topology on X , for which all the imbeddings $X_B \rightarrow X$ are continuous, where B runs over the system of bounded absolutely convex subsets in X , and X_B is a normed space with the unit ball B (see [19, Chapter I, Lemma 4.2]). \square

Example 1.6. Saturation X^Δ of a pseudocomplete locally convex space X is an imprint in the category LCS of locally convex spaces in its object X by means of the subcategory Smi of the Smith spaces (see definitions in [2]):

$$X^\Delta = \text{Imp}_{\text{Smi}}^{\text{LCS}} X$$

(c) Functorial properties of envelopes and imprints.

Apparently, in the general case the operations of taking envelopes and imprints are not functors. But in two important special cases this property is fulfilled.

Envelope in a class of objects with respect to the same class of objects. An envelope becomes a functor if the class of test morphisms and the class of realizing morphisms coincide, $\Phi = \Omega$, and are the class of all morphisms with ranges in some given class of objects \mathbf{L} (this is the special case of the situation described on page 26, where we require in addition that $\mathbf{L} = \mathbf{M}$).

Theorem 1.9. *Suppose that a category \mathbf{K} is co-well-powered, a class \mathbf{L} of objects differs morphisms on the outside, and each object X in \mathbf{K} has an envelope $\text{Env}^{\mathbf{L}} X$ in the class of objects \mathbf{L} (with respect to the same class of objects \mathbf{L}).*

- (i) *for each morphism $\alpha : X \rightarrow Y$ in \mathbf{K} and for each choice of envelopes $\text{env}^{\mathbf{L}} X$ and $\text{env}^{\mathbf{L}} Y$ there exists a unique morphism $\text{env}^{\mathbf{L}} \alpha : \text{Env}^{\mathbf{L}} X \rightarrow \text{Env}^{\mathbf{L}} Y$ such that the following diagram is commutative:*

$$\begin{array}{ccc} X & \xrightarrow{\text{env}^{\mathbf{L}} X} & \text{Env}^{\mathbf{L}} X \\ \downarrow \alpha & & \downarrow \text{env}^{\mathbf{L}} \alpha \\ Y & \xrightarrow{\text{env}^{\mathbf{L}} Y} & \text{Env}^{\mathbf{L}} Y \end{array} \quad (1.48)$$

- (ii) *the correspondence $(X, \alpha) \mapsto (\text{Env}^{\mathbf{L}} X, \text{env}^{\mathbf{L}} \alpha)$ can be defined as a covariant functor from \mathbf{K} into \mathbf{K} :*

$$\text{env}^{\mathbf{L}} 1_X = 1_{\text{Env}^{\mathbf{L}} X}, \quad \text{env}^{\mathbf{L}} (\beta \circ \alpha) = \text{env}^{\mathbf{L}} \beta \circ \text{env}^{\mathbf{L}} \alpha.$$

Proof. From the fact that \mathbf{L} differs morphisms on the outside it follows by Theorem 1.4 that the envelope $\text{env}^{\mathbf{L}} X : X \rightarrow \text{Env}^{\mathbf{L}} X$ is a bimorphism, and as a corollary, an epimorphism. Since \mathbf{K} is co-well-powered, the correspondence $X \mapsto \text{Env}^{\mathbf{L}} X$ can be defined as a map (with the help of the choice axiom). The existence of morphism $\text{env}^{\mathbf{L}} \alpha$ follows from the fact that the composition $\text{env}^{\mathbf{L}} Y \circ \alpha$ acts into the object $\text{Env}^{\mathbf{L}} Y$ which lies in \mathbf{L} . This morphism is unique, since $\text{env}^{\mathbf{L}} X$ is an extension (or because it is an epimorphism). This implies, first, that the correspondence $\alpha \mapsto \text{env}^{\mathbf{L}} \alpha$ can also be defined as a map, and, second, that it is a functor. \square

Imprint of a class of object by means of the same class of objects. In the dual situation the imprint becomes a functor, if the class of test morphisms coincide with the class of realizing morphisms, $\Phi = \Omega$, and they are the class of morphisms from a given class of objects L (this is a special case of the definition on page 32, where in addition $L = M$).

Theorem 1.10. *Suppose that a category K is well-powered, a class L of objects differs morphisms on the inside, and in each object X in K there is an imprint $\text{Imp}^L X$ of the class of objects L (by means of the same class of objects L). Then*

- (i) *for each morphism $\alpha : X \rightarrow Y$ in K and for each choice of imprints $\text{Imp}^L X$ and $\text{Imp}^L Y$ there exists a unique morphism $\text{imp}^L \alpha : \text{Imp}^L X \rightarrow \text{Imp}^L Y$ such that the following diagram is commutative:*

$$\begin{array}{ccc} X & \xleftarrow{\text{imp}^L X} & \text{Imp}^L X \\ \downarrow \alpha & & \downarrow \text{imp}^L \alpha \\ Y & \xleftarrow{\text{imp}^L Y} & \text{Imp}^L Y \end{array} \quad (1.49)$$

- (ii) *the correspondence $(X, \alpha) \mapsto (\text{Imp}^L X, \text{imp}^L \alpha)$ can be defined as a covariant functor from K into K :*

$$\text{imp}^L 1_X = 1_{\text{Imp}^L X}, \quad \text{imp}^L(\beta \circ \alpha) = \text{imp}^L \beta \circ \text{imp}^L \alpha.$$

Nets of epimorphisms.

- Suppose that to each object $X \in \text{Ob}(K)$ in a category K it is assigned a subset \mathcal{N}_X in the class $\text{Epi}(X)$ of all epimorphisms of the category K , going from X , and the following three requirements are fulfilled:

- (a) for each object X the set \mathcal{N}_X is non-empty and is directed to the left with respect to the pre-order (0.12) inherited from $\text{Epi}(X)$:

$$\forall \sigma, \sigma' \in \mathcal{N}_X \quad \exists \rho \in \mathcal{N}_X \quad \rho \rightarrow \sigma \text{ \& } \rho \rightarrow \sigma',$$

- (b) for each object X the covariant system of morphisms generated by \mathcal{N}_X

$$\text{Bind}(\mathcal{N}_X) := \{\iota_\rho^\sigma; \rho, \sigma \in \mathcal{N}_X, \rho \rightarrow \sigma\} \quad (1.50)$$

(the morphisms ι_ρ^σ were defined in (0.13); by (0.14) this system is a covariant functor from the set \mathcal{N}_X considered as a full subcategory in $\text{Epi}(X)$ into K) has a projective limit in K ;

- (c) for each morphism $\alpha : X \rightarrow Y$ and for each element $\tau \in \mathcal{N}_Y$ there are an element $\sigma \in \mathcal{N}_X$ and a morphism $\alpha_\sigma^\tau : \text{Ran}(\sigma) \rightarrow \text{Ran}(\tau)$ such that the following diagram is commutative

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ \sigma \downarrow & & \downarrow \tau \\ \text{Ran}(\sigma) & \xrightarrow{\alpha_\sigma^\tau} & \text{Ran}(\tau) \end{array} \quad (1.51)$$

(a remark: for given α , σ and τ the morphism α_σ^τ , if exists, must be unique, since σ is an epimorphism).

Then

- we call the family of set $\mathcal{N} = \{\mathcal{N}_X; X \in \text{Ob}(K)\}$ a *net of epimorphisms* in the category K , and the elements of the sets \mathcal{N}_X *elements of the net \mathcal{N}* ,
- for each object X the system of morphisms $\text{Bind}(\mathcal{N}_X)$ defined by equalities (1.50) will be called the *system of binding morphisms of the net \mathcal{N} over the vertex X* , its projective limit (which exists by condition (b)) is a projective cone, whose vertex will be denoted by $X_{\mathcal{N}}$, and the morphisms going from it by $\sigma_{\mathcal{N}} = \varprojlim_{\rho \in \mathcal{N}_X} \iota_\rho^\sigma : X_{\mathcal{N}} \rightarrow \text{Ran}(\sigma)$:

$$\begin{array}{ccc} & X_{\mathcal{N}} & \\ \rho_{\mathcal{N}} \swarrow & & \searrow \sigma_{\mathcal{N}} \\ \text{Ran}(\rho) & \xrightarrow{\iota_\rho^\sigma} & \text{Ran}(\sigma) \end{array} \quad (\rho \rightarrow \sigma); \quad (1.52)$$

in addition, by (0.13), the system of epimorphisms \mathcal{N}_X is also a projective cone of the system $\text{Bind}(\mathcal{N}_X)$:

$$\begin{array}{ccc} & X & \\ \rho \swarrow & & \searrow \sigma \\ \text{Ran}(\rho) & \xrightarrow{\iota_\rho^\sigma} & \text{Ran}(\sigma) \end{array} \quad (\rho \rightarrow \sigma), \quad (1.53)$$

so there must exist a natural morphism from X into the vertex $X_{\mathcal{N}}$ of the projective limit of the system $\text{Bind}(\mathcal{N}_X)$. We denote this morphism by $\varprojlim \mathcal{N}_X$ and call it the *local limit of the net \mathcal{N} of epimorphisms at the object X* :

$$\begin{array}{ccc} X & \xrightarrow{\varprojlim \mathcal{N}_X} & X_{\mathcal{N}} \\ \sigma \searrow & & \swarrow \sigma_{\mathcal{N}} \\ & \text{Ran}(\sigma) & \end{array} \quad (\sigma \in \mathcal{N}_X). \quad (1.54)$$

— the element σ of the net in diagram (1.51) will be called a *counterfort* of the element τ of the net.

Theorem 1.11. *If \mathcal{N} is a net of epimorphisms in a category \mathbf{K} , then for each morphism $\alpha : X \rightarrow Y$ in \mathbf{K} and for each choice of local limits $\varprojlim \mathcal{N}_X$ and $\varprojlim \mathcal{N}_Y$ the formula*

$$\alpha_{\mathcal{N}} = \varprojlim_{\tau \in \mathcal{N}_Y} \varprojlim_{\sigma \in \mathcal{N}_X} \alpha_\sigma^\tau \circ \sigma_{\mathcal{N}} \quad (1.55)$$

defines a morphism $\alpha_{\mathcal{N}} : X_{\mathcal{N}} \rightarrow Y_{\mathcal{N}}$ such that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\varprojlim \mathcal{N}_X} & X_{\mathcal{N}} \\ \alpha \downarrow & & \downarrow \alpha_{\mathcal{N}} \\ Y & \xrightarrow{\varprojlim \mathcal{N}_Y} & Y_{\mathcal{N}} \end{array}, \quad (1.56)$$

If moreover, \mathbf{K} is projectively determined⁵, then the correspondence $(X, \alpha) \mapsto (X_{\mathcal{N}}, \alpha_{\mathcal{N}})$ can be defined as a covariant functor from \mathbf{K} into \mathbf{K} :

$$(1_X)_{\mathcal{N}} = 1_{X_{\mathcal{N}}}, \quad (\beta \circ \alpha)_{\mathcal{N}} = \beta_{\mathcal{N}} \circ \alpha_{\mathcal{N}}. \quad (1.57)$$

Proof. 1. Let us explain first the sense of formula (1.55). Take a morphism $\alpha : X \rightarrow Y$. For each element $\tau \in \mathcal{N}_Y$ of the net denote

$$\alpha^\tau = \tau \circ \alpha. \quad (1.58)$$

Clearly, the family of morphisms $\{\alpha^\tau : X \rightarrow \text{Ran}(\tau); \tau \in \mathcal{N}_Y\}$ is a projective cone of the system of binding morphisms $\text{Bind}(\mathcal{N}_Y)$:

$$\begin{array}{ccc} & X & \\ \alpha^\tau \swarrow & & \searrow \alpha^v \\ \text{Ran}(\tau) & \xrightarrow{\iota_\tau^v} & \text{Ran}(v) \end{array} \quad (\tau \rightarrow v). \quad (1.59)$$

By property (c) for each element $\tau \in \mathcal{N}_Y$ there are an element $\sigma \in \mathcal{N}_X$ and a morphism $\alpha_\sigma^\tau : \text{Ran}(\sigma) \rightarrow \text{Ran}(\tau)$ such that diagram (1.51) is commutative, and we have already denoted by α^τ the diagonal there:

$$\alpha^\tau = \tau \circ \alpha = \alpha_\sigma^\tau \circ \sigma. \quad (1.60)$$

Put

$$\alpha_{\mathcal{N}}^\tau = \alpha_\sigma^\tau \circ \sigma_{\mathcal{N}}, \quad (1.61)$$

⁵Projectively determined categories were defined on p.3.

then we obtain a diagram

$$\begin{array}{ccc}
 X & \xrightarrow{\varprojlim \mathcal{N}_X} & X_{\mathcal{N}} \\
 \sigma \searrow & & \swarrow \sigma_{\mathcal{N}} \\
 & \text{Ran}(\sigma) & \\
 \alpha^{\tau} \searrow & \downarrow \alpha_{\sigma}^{\tau} & \swarrow \alpha_{\mathcal{N}}^{\tau} \\
 & \text{Ran}(\tau) &
 \end{array} \quad (\sigma \in \mathcal{N}_X). \quad (1.62)$$

Note then that for any other element $\rho \in \mathcal{N}_X$ such that $\rho \rightarrow \sigma$ the following equality analogous to (1.61) is true:

$$\alpha_{\mathcal{N}}^{\tau} = \alpha_{\rho}^{\tau} \circ \rho_{\mathcal{N}}, \quad \rho \rightarrow \sigma. \quad (1.63)$$

Indeed, for $\rho \rightarrow \sigma$ diagram (1.51) can be added to the diagram

$$\begin{array}{ccccc}
 & & X & \xrightarrow{\alpha} & Y \\
 & \nearrow \rho & \downarrow \sigma & & \downarrow \tau \\
 & & \text{Ran}(\sigma) & \xrightarrow{\alpha_{\sigma}^{\tau}} & \text{Ran}(\tau) \\
 & \nearrow \iota_{\rho}^{\sigma} & & \nearrow \alpha_{\rho}^{\tau} & \\
 \text{Ran}(\rho) & \xrightarrow{\quad} & & &
 \end{array} \quad (1.64)$$

(here the dotted arrow is initially defined as composition $\alpha_{\sigma}^{\tau} \circ \iota_{\rho}^{\sigma}$, and then, since such an arrow if it exists is unique, we deduce that this is the morphism α_{ρ}^{τ}). After that we have:

$$\alpha_{\mathcal{N}}^{\tau} = \alpha_{\sigma}^{\tau} \circ \sigma_{\mathcal{N}} = (1.52) = \alpha_{\sigma}^{\tau} \circ \iota_{\rho}^{\sigma} \circ \rho_{\mathcal{N}} = (1.64) = \alpha_{\rho}^{\tau} \circ \rho_{\mathcal{N}}.$$

From (1.63) it follows that the definition of $\alpha_{\mathcal{N}}^{\tau}$ by (1.61) does not depend on the choice of element $\sigma \in \mathcal{N}_X$, since if $\sigma' \in \mathcal{N}_X$ is another element such that there exists a morphism $\alpha_{\sigma'}^{\tau} : \text{Ran}(\sigma') \rightarrow \text{Ran}(\tau)$ for which diagram (1.51) is commutative (where σ is replaced by σ'), then we can take $\rho \in \mathcal{N}_X$ standing from the left of σ and σ' ,

$$\rho \rightarrow \sigma, \quad \rho \rightarrow \sigma',$$

(at this moment we use Axiom (a) of the net of epimorphisms) and we have the chain

$$\alpha_{\mathcal{N}}^{\tau} = \alpha_{\sigma}^{\tau} \circ \sigma_{\mathcal{N}} = (1.63) = \alpha_{\rho}^{\tau} \circ \rho_{\mathcal{N}} = (1.63) = \alpha_{\sigma'}^{\tau} \circ \sigma'_{\mathcal{N}}.$$

We can deduce now that formula (1.61) correctly defines a map $\tau \in \mathcal{N}_Y \mapsto \alpha_{\mathcal{N}}^{\tau}$. Let us show that the family of morphisms $\{\alpha_{\mathcal{N}}^{\tau} : X_{\mathcal{N}} \rightarrow \text{Ran}(\tau); \tau \in \mathcal{N}_Y\}$ is a projective cone of the system of binding morphisms $\text{Bind}(\mathcal{N}_Y)$.

$$\begin{array}{ccc}
 & X_{\mathcal{N}} & \\
 \alpha_{\mathcal{N}}^{\tau} \swarrow & & \searrow \alpha_{\mathcal{N}}^v \\
 \text{Ran}(\tau) & \xrightarrow{\iota_{\tau}^v} & \text{Ran}(v)
 \end{array} \quad (\tau \rightarrow v \in \mathcal{N}_Y). \quad (1.65)$$

For $\tau \rightarrow v$ diagram (1.51) can be added to the diagram

$$\begin{array}{ccccc}
 & & X & \xrightarrow{\alpha} & Y \\
 & \nearrow \sigma & \downarrow \sigma & & \downarrow \tau \\
 & & \text{Ran}(\sigma) & \xrightarrow{\alpha_{\sigma}^{\tau}} & \text{Ran}(\tau) \\
 & \nearrow \alpha_{\sigma}^v & & \nearrow \iota_{\tau}^v & \\
 \text{Ran}(\rho) & \xrightarrow{\quad} & & &
 \end{array} \quad (1.66)$$

(where the dotted arrow is initially defined as the composition $\iota_\tau^v \circ \alpha_\sigma^\tau$, and then, since such an arrow, if it exists, is unique, we deduce that this is the morphism α_σ^v). Using this diagram we have:

$$\iota_\tau^v \circ \alpha_\mathcal{N}^\tau = (1.61) = \iota_\tau^v \circ \alpha_\sigma^\tau \circ \sigma_\mathcal{N} = (1.66) = \alpha_\sigma^v \circ \sigma_\mathcal{N} = (1.61) = \alpha_\mathcal{N}^v.$$

From diagram (1.65) it follows now that there must exist a natural morphism $\alpha_\mathcal{N}$ from $X_\mathcal{N}$ into the projective limit $Y_\mathcal{N}$ of the system $\text{Bind}(\mathcal{N}_Y)$:

$$\begin{array}{ccc} X_\mathcal{N} & \xrightarrow{\alpha_\mathcal{N}} & Y_\mathcal{N} \\ \alpha_\mathcal{N}^\tau \searrow & & \swarrow \tau_\mathcal{N} \\ & \text{Ran}(\tau) & \end{array} \quad (\tau \in \mathcal{N}_Y). \quad (1.67)$$

Recall now that by Axiom (b) of the net the passage from X to the projective limit $\varprojlim \text{Bind}(\mathcal{N}_X)$ can be organized as a map. The further steps on building $\alpha_\mathcal{N}$ (the choice of the vertex $X_\mathcal{N}$ of the cone $\varprojlim \text{Bind}(\mathcal{N}_X)$, and then the choice of the arrow $\alpha_\mathcal{N}$ such that all the diagrams (1.67) are commutative) is also unambiguous, so the correspondence $\alpha \mapsto \alpha_\mathcal{N}$ can also be treated as a map.

2. Note then that for the morphisms $\alpha_\mathcal{N}$ the diagrams of the form (1.56) are commutative. In the diagram

$$\begin{array}{ccccc} X & & \xrightarrow{\varprojlim \mathcal{N}_X} & & X_\mathcal{N} \\ & \searrow \alpha^\tau & & \swarrow \alpha_\mathcal{N}^\tau & \\ & & \text{Ran}(\tau) & & \\ & \swarrow \tau & & \searrow \tau_\mathcal{N} & \\ Y & & \xrightarrow{\varprojlim \mathcal{N}_Y} & & Y_\mathcal{N} \end{array}$$

α (left vertical), $\alpha_\mathcal{N}$ (right vertical), $\varprojlim \mathcal{N}_X$ (top horizontal), $\varprojlim \mathcal{N}_Y$ (bottom horizontal).

all the inner triangles are commutative: the upper inner triangle is commutative because this is the perimeter of (1.62), the left inner triangle because this is a variant of formula (1.58), the lower inner triangle because this is up to notations diagram (1.54), and the right inner triangle because this is a rotated diagram (1.67). So the following equalities are true:

$$\tau_\mathcal{N} \circ \varprojlim \mathcal{N}_Y \circ \alpha = \alpha^\tau = \tau_\mathcal{N} \circ \alpha_\mathcal{N} \circ \varprojlim \mathcal{N}_X \quad (\tau \in \mathcal{N}_Y)$$

One can interpret this as follows: each of the morphisms $\varprojlim \mathcal{N}_Y \circ \alpha$ and $\alpha_\mathcal{N} \circ \varprojlim \mathcal{N}_X$ is a lifting of the projective cone $\{\alpha^\tau : X \rightarrow \text{Ran}(\tau); \tau \in \mathcal{N}_Y\}$ for the system of binding morphisms $\text{Bind}(\mathcal{N}_Y)$ which we were talking about in diagram (1.59) to the projective limit of this system. I.e. $\varprojlim \mathcal{N}_Y \circ \alpha$ and $\alpha_\mathcal{N} \circ \varprojlim \mathcal{N}_X$ are the very same dotted arrow in the definition of projective limit, for which all the diagrams of the form

$$\begin{array}{ccc} X & \xrightarrow{\quad} & Y_\mathcal{N} \\ \alpha^\tau \searrow & & \swarrow \tau_\mathcal{N} \\ & \text{Ran}(\tau) & \end{array} \quad (\tau \in \mathcal{N}_Y).$$

are commutative. But such an arrow is unique, so these morphisms must coincide:

$$\varprojlim \mathcal{N}_Y \circ \alpha = \alpha_\mathcal{N} \circ \varprojlim \mathcal{N}_X.$$

This is exactly diagram (1.56).

3. Suppose now that \mathbb{K} is projectively determined. Then the operation of taking this limit can be organized as a map:

$$X \mapsto \varprojlim \text{Bind}(\mathcal{N}_X)$$

(i.e. there is a map which assigns to each object $X \in \text{Ob}(\mathbb{K})$ a concrete projective limit of the subcategory $\text{Bind}(\mathcal{N}_X)$ among all its projective limits in \mathbb{K}). Let us show that in this case the arising map $(X, \alpha) \mapsto (X_\mathcal{N}, \alpha_\mathcal{N})$ is a functor, i.e. the identities (1.57) are fulfilled. Suppose first that $\alpha = 1_X : X \rightarrow X$. Then

$$\begin{aligned} \alpha^\tau = (1.58) = \tau \circ \alpha = \tau \circ 1_X = \tau &\implies \alpha_\sigma^\tau \circ \sigma = (1.60) = \alpha^\tau = \tau = (0.13) = \iota_\sigma^\tau \circ \sigma \implies \\ &\implies \alpha_\sigma^\tau = \iota_\sigma^\tau \implies \alpha_\mathcal{N}^\tau = \iota_\sigma^\tau \circ \sigma_\mathcal{N} = \tau_\mathcal{N} \end{aligned}$$

So in diagrams (1.67) we can replace $\alpha_{\mathcal{N}}^{\tau}$ by $\tau_{\mathcal{N}}$:

$$\begin{array}{ccc} X_{\mathcal{N}} & \overset{\alpha_{\mathcal{N}}}{\dashrightarrow} & X_{\mathcal{N}} \\ & \searrow \tau_{\mathcal{N}} \quad \swarrow \tau_{\mathcal{N}} & \\ & \text{Ran}(\tau) & \end{array} \quad (\tau \in \mathcal{N}_X).$$

These diagrams are commutative for all $\tau \in \mathcal{N}_X$, and the dotted arrow $\alpha_{\mathcal{N}}$ is defined here as the lifting of the projective cone $\{\alpha_{\mathcal{N}}^{\tau} = \tau_{\mathcal{N}} : X_{\mathcal{N}} \rightarrow \text{Ran}(\tau)\}$ to the projective limit $\{\tau_{\mathcal{N}} : X_{\mathcal{N}} \rightarrow \text{Ran}(\tau)\}$. Such an arrow is unique, so it must coincide with the morphism $1_{X_{\mathcal{N}}}$, for which all these diagrams are trivially commutative: $\alpha_{\mathcal{N}} = 1_{X_{\mathcal{N}}}$.

Let us now prove the second identity in (1.57). Consider the sequence of morphisms $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$. Take an element $v \in \mathcal{N}_Z$ and, using Axiom (c), let us choose first an element $\tau \in \mathcal{N}_Y$ and a morphism β_{τ}^v such that

$$v \circ \beta = \beta_{\tau}^v \circ \tau,$$

And then again using Axiom (c) choose an element $\sigma \in \mathcal{N}_X$ and a morphism α_{σ}^{τ} such that

$$\tau \circ \alpha = \alpha_{\sigma}^{\tau} \circ \sigma.$$

We obtain the following diagram:

$$\begin{array}{ccccc} X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z \\ \downarrow \sigma & & \downarrow \tau & & \downarrow v \\ \text{Ran}(\sigma) & \xrightarrow{\alpha_{\sigma}^{\tau}} & \text{Ran}(\tau) & \xrightarrow{\beta_{\tau}^v} & \text{Ran}(v) \end{array}.$$

If we remove here the middle arrow, then we obtain a diagram

$$\begin{array}{ccc} X & \xrightarrow{\beta \circ \alpha} & Z \\ \downarrow \sigma & & \downarrow v \\ \text{Ran}(\sigma) & \xrightarrow{\beta_{\tau}^v \circ \alpha_{\sigma}^{\tau}} & \text{Ran}(v) \end{array},$$

which can be understood in such a way that the morphism $\beta_{\tau}^v \circ \alpha_{\sigma}^{\tau}$ is exactly the unique dotted arrow from diagram (1.51), but the difference is that Y is replaced here by Z , α by $\beta \circ \alpha$, and τ by v . Hence we can deduce that there exists a morphism $(\beta \circ \alpha)_{\sigma}^v$ which coincide with $\beta_{\tau}^v \circ \alpha_{\sigma}^{\tau}$:

$$\beta_{\tau}^v \circ \alpha_{\sigma}^{\tau} = (\beta \circ \alpha)_{\sigma}^v \quad (1.68)$$

This equality is used in the following chain:

$$\underbrace{v_{\mathcal{N}} \circ \beta_{\mathcal{N}}}_{\parallel (1.67)} \circ \alpha_{\mathcal{N}} = \beta_{\tau}^v \circ \underbrace{\tau_{\mathcal{N}} \circ \alpha_{\mathcal{N}}}_{\parallel (1.67)} = \underbrace{\beta_{\tau}^v \circ \alpha_{\sigma}^{\tau}}_{\parallel (1.68)} \circ \sigma_{\mathcal{N}} = \underbrace{(\beta \circ \alpha)_{\sigma}^v \circ \sigma_{\mathcal{N}}}_{\parallel (1.61)} = (\beta \circ \alpha)_{\mathcal{N}}^v = (1.67) = v_{\mathcal{N}} \circ (\beta \circ \alpha)_{\mathcal{N}}.$$

$\parallel (1.61)$
 $\beta_{\mathcal{N}}^v$
 $\parallel (1.61)$
 $\beta_{\tau}^v \circ \tau_{\mathcal{N}}$

If we omit the intermediate calculations, we arrive at the following double equality:

$$v_{\mathcal{N}} \circ (\beta_{\mathcal{N}} \circ \alpha_{\mathcal{N}}) = (\beta \circ \alpha)_{\mathcal{N}}^v = v_{\mathcal{N}} \circ (\beta \circ \alpha)_{\mathcal{N}}.$$

This is true for each $v \in \mathcal{N}_Z$. So this can be treated as if both $\beta_{\mathcal{N}} \circ \alpha_{\mathcal{N}}$ and $(\beta \circ \tau)_{\mathcal{N}}$ were liftings of the projective cone $\{(\beta \circ \alpha)_{\mathcal{N}}^v : X_{\mathcal{N}} \rightarrow \text{Ran}(v); v \in \mathcal{N}_Z\}$ for the system of binding morphisms $\text{Bind}(\mathcal{N}_Z)$ (and this family is indeed a projective cone due to diagram (1.65) where one should replace Y by Z , and α by $\beta \circ \alpha$) to the projective limit of this system. Thus, $\beta_{\mathcal{N}} \circ \alpha_{\mathcal{N}}$ and $(\beta \circ \tau)_{\mathcal{N}}$ are exactly the dotted arrow in the definition of projective limit, for which all the diagrams of the form

$$\begin{array}{ccc} X_{\mathcal{N}} & \dashrightarrow & Z_{\mathcal{N}} \\ & \searrow (\beta \circ \alpha)_{\mathcal{N}}^v \quad \swarrow v_{\mathcal{N}} & \\ & \text{Ran}(v) & \end{array} \quad (v \in \mathcal{N}_Z).$$

are commutative. But this dotted arrow is unique, so these morphisms must coincide:

$$\beta_{\mathcal{N}} \circ \alpha_{\mathcal{N}} = (\beta \circ \tau)_{\mathcal{N}}.$$

This is the identity (1.57). \square

Theorem 1.12. *Suppose \mathcal{N} is a net of epimorphisms in a category \mathbf{K} . Then*

- (a) *for each object X in \mathbf{K} the local limit $\varprojlim \mathcal{N}_X$ is an envelope $\text{env}_{\mathcal{N}} X$ in the category \mathbf{K} with respect to the class of morphisms \mathcal{N} :*

$$\varprojlim \mathcal{N}_X = \text{env}_{\mathcal{N}} X, \quad (1.69)$$

- (b) *if \mathbf{K} is projectively determined, then there is a covariant functor $(X, \alpha) \mapsto (\text{Env}_{\mathcal{N}} X, \text{env}_{\mathcal{N}} \alpha)$ from \mathbf{K} into \mathbf{K} ,*

$$\text{env}_{\mathcal{N}}(1_X) = 1_{\text{Env}_{\mathcal{N}} X}, \quad \text{env}_{\mathcal{N}}(\beta \circ \alpha) = \text{env}_{\mathcal{N}} \beta \circ \text{env}_{\mathcal{N}} \alpha, \quad (1.70)$$

such that for any morphism $\alpha : X \rightarrow Y$ in \mathbf{K} the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\text{env}_{\mathcal{N}} X} & \text{Env}_{\mathcal{N}} X \\ \downarrow \alpha & & \downarrow \text{env}_{\mathcal{N}} \alpha \\ Y & \xrightarrow{\text{env}_{\mathcal{N}} Y} & \text{Env}_{\mathcal{N}} Y \end{array} \quad (1.71)$$

Proof. By Lemma 1.1 the projective limit $\varprojlim \mathcal{N}_X$ is an envelope of X in \mathbf{K} with respect to the cone of morphisms \mathcal{N}_X :

$$\varprojlim \mathcal{N}_X = \text{env}_{\mathcal{N}_X} X$$

In the last expression one can replace \mathcal{N}_X by \mathcal{N} , since \mathcal{N}_X is exactly a subclass in \mathcal{N} consisting of morphisms which go from X :

$$\varprojlim \mathcal{N}_X = \text{env}_{\mathcal{N}_X} X = \text{env}_{\mathcal{N}} X.$$

The rest follows from Theorem 1.11. \square

Theorem 1.13. *Suppose in a category \mathbf{K} we have a net of epimorphisms \mathcal{N} and classes of morphisms Ω and Φ , such that the following conditions are fulfilled:*

- (i) *all the local limits $\varprojlim \mathcal{N}_X$ belong to Ω , and Ω lies in the class of all epimorphisms of the category \mathbf{K} :*

$$\{\varprojlim \mathcal{N}_X; X \in \text{Ob}(\mathbf{K})\} \subseteq \Omega \subseteq \text{Epi}(\mathbf{K}),$$

- (ii) *the net \mathcal{N} generates the class Φ on the inside:*

$$\mathcal{N} \subseteq \Phi \subseteq \text{Mor}(\mathbf{K}) \circ \mathcal{N}.$$

Then

- (a) *for each object X in \mathbf{K} the local limit $\varprojlim \mathcal{N}_X$ is an envelope $\text{env}_{\Phi}^{\Omega} X$ in the class Ω with respect to the class Φ :*

$$\varprojlim \mathcal{N}_X = \text{env}_{\Phi}^{\Omega} X, \quad (1.72)$$

- (b) *if \mathbf{K} is projectively determined, then there is a covariant functor $(X, \alpha) \mapsto (\text{Env}_{\Phi}^{\Omega} X, \text{env}_{\Phi}^{\Omega} \alpha)$ from \mathbf{K} into \mathbf{K} ,*

$$\text{env}_{\Phi}^{\Omega}(1_X) = 1_{\text{Env}_{\Phi}^{\Omega} X}, \quad \text{env}_{\Phi}^{\Omega}(\beta \circ \alpha) = \text{env}_{\Phi}^{\Omega} \beta \circ \text{env}_{\Phi}^{\Omega} \alpha, \quad (1.73)$$

such that for any morphism $\alpha : X \rightarrow Y$ in \mathbf{K} the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\text{env}_{\Phi}^{\Omega} X} & \text{Env}_{\Phi}^{\Omega} X \\ \downarrow \alpha & & \downarrow \text{env}_{\Phi}^{\Omega} \alpha \\ Y & \xrightarrow{\text{env}_{\Phi}^{\Omega} Y} & \text{Env}_{\Phi}^{\Omega} Y \end{array} \quad (1.74)$$

Proof. 1. By Theorem 1.12 the local limit of the net $\varprojlim \mathcal{N}_X$ is an envelope of X in the class $\mathbf{Mor}(\mathbf{K})$ of all morphisms of the category \mathbf{K} with respect to the class of morphisms \mathcal{N} :

$$\varprojlim \mathcal{N}_X = \mathbf{env}_{\mathcal{N}} X := \mathbf{env}_{\mathcal{N}}^{\mathbf{Mor}(\mathbf{K})} X.$$

On the other hand, by (i) $\varprojlim \mathcal{N}_X$ belongs to a narrower class Ω , so by 1° (c) on page 20, $\varprojlim \mathcal{N}_X$ must be an envelope in this narrower class Ω :

$$\varprojlim \mathcal{N}_X = \mathbf{env}_{\mathcal{N}} X = \mathbf{env}_{\mathcal{N}}^{\mathbf{Mor}(\mathbf{K})} X = \mathbf{env}_{\mathcal{N}}^{\Omega} X.$$

Further, since \mathcal{N} generates Φ on the inside, and Ω consists of epimorphisms, by (1.14) the envelope with respect to \mathcal{N} must coincide with the envelope with respect to Φ :

$$\varprojlim \mathcal{N}_X = \mathbf{env}_{\mathcal{N}} X = \mathbf{env}_{\mathcal{N}}^{\mathbf{Mor}(\mathbf{K})} X = \mathbf{env}_{\mathcal{N}}^{\Omega} X = \mathbf{env}_{\Phi}^{\Omega} X.$$

This proves (1.72).

2. The existence of the dotted arrow in (1.74) follows now from Theorem 1.12 (or from Theorem 1.11), and its uniqueness – from the epimorphy of the morphism $\varprojlim \mathcal{N}_X = \mathbf{env}_{\Phi}^{\Omega} X$. Finally, (1.73) follows from (1.70) (and from the uniqueness of the dotted arrow in (1.74)). \square

The requirement of Theorem 1.13 that the net of epimorphisms \mathcal{N} must consist of epimorphisms of the category \mathbf{K} can be replaced by another condition which is described with the help of the following definition (however, it is not clear, whether this definition is informative, since the author does not know any examples of the nets with this property – see below counterexample 4.3).

- Let us say that in diagram (1.51)

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ \sigma \downarrow & & \downarrow \tau \\ \mathbf{Ran}(\sigma) & \xrightarrow{\alpha_{\sigma}^{\tau}} & \mathbf{Ran}(\tau) \end{array}$$

the counterfort σ of an element τ is *relatively splitted*, if for each morphism $\delta : Y \rightarrow \mathbf{Ran}(\sigma)$ the commutativity of the diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ \sigma \downarrow & \swarrow \delta & \\ \mathbf{Ran}(\sigma) & & \end{array}$$

automatically implies the commutativity of the diagram

$$\begin{array}{ccc} & & Y \\ & \swarrow \delta & \downarrow \tau \\ \mathbf{Ran}(\sigma) & \xrightarrow{\alpha_{\sigma}^{\tau}} & \mathbf{Ran}(\tau). \end{array}$$

If in the net of epimorphisms \mathcal{N} each element $\tau \in \mathcal{N}_Y$ has a relatively splitted counterfort for any morphism $\alpha : X \rightarrow Y$ in \mathbf{K} , then the net \mathcal{N} will be called a *relatively splitted net of epimorphisms*.

Theorem 1.14. Suppose in a category \mathbf{K} we have a relatively splitted net of epimorphisms \mathcal{N} , and classes of morphisms Ω and Φ such that:

- (i) all the local limits $\varprojlim \mathcal{N}_X$ belong to class Ω :

$$\{\varprojlim \mathcal{N}_X; X \in \mathbf{Ob}(\mathbf{K})\} \subseteq \Omega,$$

- (ii) the net \mathcal{N} generates the class Φ on the inside:

$$\mathcal{N} \subseteq \Phi \subseteq \mathbf{Mor}(\mathbf{K}) \circ \mathcal{N}.$$

Then

- (a) for each object X in \mathbf{K} the local limit $\varprojlim \mathcal{N}_X$ is an envelope $\text{env}_{\Phi}^{\Omega} X$ in the class of morphisms Ω with respect to the class of morphisms Φ :

$$\varprojlim \mathcal{N}_X = \text{env}_{\Phi}^{\Omega} X, \quad (1.75)$$

- (b) if \mathbf{K} is projectively determined, then there is a covariant functor $(X, \alpha) \mapsto (\text{Env}_{\Phi}^{\Omega} X, \text{env}_{\Phi}^{\Omega} \alpha)$ from \mathbf{K} onto \mathbf{K}

$$\text{env}_{\Phi}^{\Omega}(1_X) = 1_{\text{Env}_{\Phi}^{\Omega} X}, \quad \text{env}_{\Phi}^{\Omega}(\beta \circ \alpha) = \text{env}_{\Phi}^{\Omega} \beta \circ \text{env}_{\Phi}^{\Omega} \alpha, \quad (1.76)$$

such that for any morphism $\alpha : X \rightarrow Y$ in \mathbf{K} the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\text{env}_{\Phi}^{\Omega} X} & \text{Env}_{\Phi}^{\Omega} X \\ \downarrow \alpha & & \downarrow \text{env}_{\Phi}^{\Omega} \alpha \\ Y & \xrightarrow{\text{env}_{\Phi}^{\Omega} Y} & \text{Env}_{\Phi}^{\Omega} Y \end{array} \quad (1.77)$$

Proof. The difference between this situation and the one in Theorem 1.13 is that we cannot use here the equality (1.14).

1. By Theorem 1.12 the local limit $\varprojlim \mathcal{N}_X$ is an envelope of X in the class $\text{Mor}(\mathbf{K})$ of all morphisms in the category \mathbf{K} with respect to the class of morphisms \mathcal{N} :

$$\varprojlim \mathcal{N}_X = \text{env}_{\mathcal{N}} X := \text{env}_{\mathcal{N}}^{\text{Mor}(\mathbf{K})} X.$$

On the other hand, by (i) $\varprojlim \mathcal{N}_X$ belongs to a narrower class Ω , so by 1° (c) on page 20, $\varprojlim \mathcal{N}_X$ is an envelope in this narrower class Ω :

$$\varprojlim \mathcal{N}_X = \text{env}_{\mathcal{N}} X = \text{env}_{\mathcal{N}}^{\text{Mor}(\mathbf{K})} X = \text{env}_{\Phi}^{\Omega} X.$$

Hence for proof of (1.75) it remains to verify the equality

$$\text{env}_{\mathcal{N}}^{\Omega} X = \text{env}_{\Phi}^{\Omega} X. \quad (1.78)$$

Denote $\omega = \text{env}_{\mathcal{N}}^{\Omega} X$ and let us show first that ω is an extension of X in Ω with respect to Φ . Take $\varphi : X \rightarrow M$, $\varphi \in \Phi$. Since \mathcal{N} generates Φ on the inside, there is an element $\rho \in \mathcal{N}_X$ and a morphism $\chi : \text{Ran}(\rho) \rightarrow M$ such that

$$\varphi = \chi \circ \rho.$$

Since ω is an extension with respect to \mathcal{N} , the morphism ρ can be continued to some morphism $\rho' : \text{Env}_{\mathcal{N}}^{\Omega} X \rightarrow \text{Ran}(\rho)$, and after that we obtain that $\varphi' = \chi \circ \rho'$ is a continuation of φ at $\text{Env}_{\mathcal{N}}^{\Omega} X$:

$$\begin{array}{ccc} X & \xrightarrow{\omega} & \text{Env}_{\mathcal{N}}^{\Omega} X \\ \downarrow \rho & \searrow \rho' & \\ \text{Ran}(\rho) & & \\ \downarrow \chi & & \\ M & & \end{array} \quad \begin{array}{c} \varphi \\ \varphi' \end{array}$$

Let us show now that this continuation φ' is unique. Suppose that $\varphi', \varphi'' : \text{Env}_{\mathcal{N}}^{\Omega} X \rightarrow M$ are two continuations for φ , i.e. $\varphi' \circ \omega = \varphi = \varphi'' \circ \omega$. Our reasoning will be illustrated by the following diagram, where the left-half and the right-half are commutative separately from each other, and the numbers mean the order of construction of the arrows:

$$\begin{array}{ccccc} \text{Ran}(\sigma) & \xleftarrow{\sigma} & X & \xrightarrow{\sigma} & \text{Ran}(\sigma) \\ \downarrow \omega_{\sigma}^{\tau} & \searrow \delta & \downarrow \omega & \nearrow \delta & \downarrow \omega_{\sigma}^{\tau} \\ \text{Ran}(\tau) & \xleftarrow{\tau} & \text{Env}_{\mathcal{N}}^{\Omega} X & \xrightarrow{\tau} & \text{Ran}(\tau) \\ & & \downarrow \varphi' & \downarrow \varphi'' & \\ & & M & & \end{array} \quad \begin{array}{c} \psi' \\ \psi'' \end{array} \quad (1.79)$$

Here the arrows with the number 1 are built as follows. Since \mathcal{N} generates Φ on the inside, there are some elements $v', v'' \in \mathcal{N}_{\text{Env}_{\mathcal{N}}^{\Omega} X}$ such that $\varphi' = \chi' \circ v'$, $\varphi'' = \chi'' \circ v''$ for some χ', χ'' . Take an element $\tau \in \mathcal{N}_{\text{Env}_{\mathcal{N}}^{\Omega} X}$ which majorizes v' and v'' , i.e. $\tau \rightarrow v'$ and $\tau \rightarrow v''$. Then $v' = \iota_{\tau}^{v'} \circ \tau$ and $v'' = \iota_{\tau}^{v''} \circ \tau$, hence $\varphi' = \chi' \circ \iota_{\tau}^{v'} \circ \tau = \psi' \circ \tau$ and $\varphi'' = \chi'' \circ \iota_{\tau}^{v''} \circ \tau = \psi'' \circ \tau$, where $\psi' = \chi' \circ \iota_{\tau}^{v'}$ and $\psi'' = \chi'' \circ \iota_{\tau}^{v''}$.

Then we fix this τ and find a relatively splitted counterfort σ for it and the corresponding binding morphism ω_{σ}^{τ} (at this moment we use the fact that \mathcal{N} is relatively splitted). That is how the arrows with number 2 appear.

Finally, the arrow with number 3, i.e. δ , is a continuation of σ along the extension ω . The commutativity of the triangles which arise on the both sides of δ follows from the relative splittability of counterforts: the equality $\sigma = \delta \circ \omega$ implies the equality $\tau = \omega_{\sigma}^{\tau} \circ \delta$.

Once diagram (1.79) is constructed, we get the following chain:

$$\begin{aligned} \psi' \circ \omega_{\sigma}^{\tau} \circ \underbrace{\sigma}_{\substack{\cap \\ \text{Epi}}} &= \varphi' \circ \omega = \varphi = \varphi'' \circ \omega = \psi'' \circ \omega_{\sigma}^{\tau} \circ \underbrace{\sigma}_{\substack{\cap \\ \text{Epi}}} \implies \psi' \circ \omega_{\sigma}^{\tau} = \psi'' \circ \omega_{\sigma}^{\tau} \implies \\ &\implies \varphi' = \psi' \circ \omega_{\sigma}^{\tau} \circ \delta = \psi'' \circ \omega_{\sigma}^{\tau} \circ \delta = \varphi''. \end{aligned}$$

Thus, we understood that $\text{env}_{\mathcal{N}}^{\Omega} X$ is an extension of X in Ω with respect to Φ . If now $\sigma : X \rightarrow X'$ is another extension of X in Ω with respect to Φ , then $\sigma : X \rightarrow X'$ is an extension of X in Ω with respect to a narrower class \mathcal{N} , so there must exist a unique morphism v from X into the envelope $\text{Env}_{\mathcal{N}}^{\Omega} X$ with respect to \mathcal{N} such that

$$\begin{array}{ccc} & X & \\ \sigma \swarrow & & \searrow \text{env}_{\mathcal{N}}^{\Omega} X \\ X' & \xrightarrow{\quad v \quad} & \text{Env}_{\mathcal{N}}^{\Omega} X \end{array}$$

This means that $\text{env}_{\mathcal{N}}^{\Omega} X$ is not just an extension, but an envelope of X in Ω with respect to Φ , so (1.78) holds, and as a corollary (1.75) holds as well.

2. Now the existence of a dotted arrow in (1.77) follows from Theorem 1.12 (or from Theorem 1.11). Finally, (1.76) follows from (1.70). \square

Nets of monomorphisms.

- Suppose that to each object $X \in \text{Ob}(\mathbf{K})$ in a category \mathbf{K} it is assigned a subset \mathcal{N}_X in the class $\text{Mono}(X)$ of all monomorphisms of \mathbf{K} coming to X , and the following three requirements are fulfilled:

- (a) for each object X the set \mathcal{N}_X is non-empty and is directed to the right with respect to pre-order (0.8) inherited from $\text{Mono}(X)$:

$$\forall \rho, \rho' \in \mathcal{N}_X \quad \exists \sigma \in \mathcal{N}_X \quad \rho \rightarrow \sigma \ \& \ \rho' \rightarrow \sigma,$$

- (b) for each object X the covariant system of morphisms generated by the set \mathcal{N}_X

$$\text{Bind}(\mathcal{N}_X) := \{\varkappa_{\rho}^{\sigma}; \rho, \sigma \in \mathcal{N}_X, \rho \rightarrow \sigma\} \quad (1.80)$$

(the morphisms $\varkappa_{\rho}^{\sigma}$ were defined in (0.9); according to (0.10), this system is a covariant functor from the set \mathcal{N}_X considered as a full subcategory in $\text{Mono}(X)$ into \mathbf{K});

- (c) for each morphism $\alpha : X \rightarrow Y$ and for each element $\sigma \in \mathcal{N}_X$ there is an element $\tau \in \mathcal{N}_Y$ and a morphism $\alpha_{\sigma}^{\tau} : \text{Dom}(\sigma) \rightarrow \text{Dom}(\tau)$ such that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ \sigma \uparrow & & \uparrow \tau \\ \text{Dom}(\sigma) & \xrightarrow{\alpha_{\sigma}^{\tau}} & \text{Dom}(\tau) \end{array} \quad (1.81)$$

(a remark: for given α , σ and τ the morphism α_{σ}^{τ} , if exists, must be unique, since τ is a monomorphism).

Then

- we call the family of sets $\mathcal{N} = \{\mathcal{N}_X; X \in \text{Ob}(\mathbf{K})\}$ a *net of monomorphisms* in the category \mathbf{K} , and the elements of the sets \mathcal{N}_X *elements of the net* \mathcal{N} ,
- for each object X the system of morphisms $\text{Bind}(\mathcal{N}_X)$ defined by equalities (1.80) will be called a *system of binding morphisms of the net \mathcal{N} over the vertex X* , its injective limit (which exists by condition (b)) is an injective cone whose vertex will be denoted by $X_{\mathcal{N}}$, and the morphisms coming to it by $\rho_{\mathcal{N}} = \varinjlim_{\sigma \in \mathcal{N}_X} \kappa_{\rho}^{\sigma} : X_{\mathcal{N}} \leftarrow \text{Ran}(\sigma)$:

$$\begin{array}{ccc} & X_{\mathcal{N}} & \\ \rho_{\mathcal{N}} \nearrow & & \nwarrow \sigma_{\mathcal{N}} \\ \text{Dom}(\rho) & \xrightarrow{\kappa_{\rho}^{\sigma}} & \text{Dom}(\sigma) \end{array} \quad (\rho \rightarrow \sigma); \quad (1.82)$$

in addition, by (0.9), the system of monomorphisms \mathcal{N}_X is also called an injective cone of the system $\text{Bind}(\mathcal{N}_X)$:

$$\begin{array}{ccc} & X & \\ \rho \nearrow & & \nwarrow \sigma \\ \text{Dom}(\rho) & \xrightarrow{\kappa_{\rho}^{\sigma}} & \text{Dom}(\sigma) \end{array} \quad (\rho \rightarrow \sigma), \quad (1.83)$$

so there must exist a natural morphism into X from the vertex $X_{\mathcal{N}}$ of the injective limit of the system $\text{Bind}(\mathcal{N}_X)$. This morphism will be denoted by $\varinjlim \mathcal{N}_X$ and will be called a *local limit of the net of monomorphisms \mathcal{N} at the object X* :

$$\begin{array}{ccc} X_{\mathcal{N}} & \xrightarrow{\varinjlim \mathcal{N}_X} & X \\ \sigma_{\mathcal{N}} \nwarrow & & \nearrow \sigma \\ & \text{Dom}(\sigma) & \end{array} \quad (\sigma \in \mathcal{N}_X). \quad (1.84)$$

- the element τ of the net in diagram (1.81) will be called a *shed* for the element σ of the net.

The following four propositions are dual to Theorems 1.11, 1.12, 1.13 and 1.14.

Theorem 1.15. *If \mathcal{N} is a net of monomorphisms in a category \mathbf{K} , then for each morphism $\alpha : X \rightarrow Y$ in \mathbf{K} and for each choice of local limits $\varinjlim \mathcal{N}_X$ and $\varinjlim \mathcal{N}_Y$ the formula*

$$\alpha_{\mathcal{N}} = \varinjlim_{\sigma \in \mathcal{N}_X} \varinjlim_{\tau \in \mathcal{N}_Y} \tau_{\mathcal{N}} \circ \alpha_{\sigma}^{\tau} \quad (1.85)$$

defines a morphism $\alpha_{\mathcal{N}} : X_{\mathcal{N}} \rightarrow Y_{\mathcal{N}}$ such that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\varinjlim \mathcal{N}_X} & X_{\mathcal{N}} \\ \downarrow \alpha & & \downarrow \alpha_{\mathcal{N}} \\ Y & \xrightarrow{\varinjlim \mathcal{N}_Y} & Y_{\mathcal{N}} \end{array}, \quad (1.86)$$

If moreover \mathbf{K} is injectively determined⁶, then the correspondence $(X, \alpha) \mapsto (X_{\mathcal{N}}, \alpha_{\mathcal{N}})$ can be defined as a covariant functor from \mathbf{K} into \mathbf{K} :

$$(1_X)_{\mathcal{N}} = 1_{X_{\mathcal{N}}}, \quad (\beta \circ \alpha)_{\mathcal{N}} = \beta_{\mathcal{N}} \circ \alpha_{\mathcal{N}}. \quad (1.87)$$

Theorem 1.16. *Suppose \mathcal{N} is a net of monomorphisms in a category \mathbf{K} . Then*

- (a) *then for each object X in \mathbf{K} the local limit $\varinjlim \mathcal{N}_X$ is an imprint $\text{imp}_{\mathbf{M}} X$ of the category \mathbf{K} in X by means of the class of morphisms \mathcal{N} :*

$$\varinjlim \mathcal{N}_X = \text{imp}_{\mathcal{N}} X, \quad (1.88)$$

⁶Injectively determined categories were defined on p.3.

(b) if \mathbf{K} is injectively determined, then there is a covariant functor $(X, \alpha) \mapsto (\text{Imp}_{\mathcal{N}} X, \text{imp}_{\mathcal{N}} \alpha)$ from \mathbf{K} into \mathbf{K}

$$\text{imp}_{\mathcal{N}}(1_X) = 1_{\text{Imp}_{\mathcal{N}} X}, \quad \text{imp}_{\mathcal{N}}(\beta \circ \alpha) = \text{imp}_{\mathcal{N}} \beta \circ \text{imp}_{\mathcal{N}} \alpha, \quad (1.89)$$

such that for any morphism $\alpha : X \rightarrow Y$ in \mathbf{K} the following diagram is commutative:

$$\begin{array}{ccc} X & \xleftarrow{\text{imp}_{\mathcal{N}} X} & \text{Imp}_{\mathcal{N}} X \\ \downarrow \alpha & & \downarrow \text{imp}_{\mathcal{N}} \alpha \\ Y & \xleftarrow{\text{imp}_{\mathcal{N}} Y} & \text{Imp}_{\mathcal{N}} Y \end{array} . \quad (1.90)$$

Theorem 1.17. Suppose that in a category \mathbf{K} we have a net of monomorphisms \mathcal{N} and classes of morphisms Ω and Φ such that the following conditions are fulfilled:

(i) all the local limits $\varinjlim \mathcal{N}_X$ belong to Ω , and Ω lies in the class of all monomorphisms of \mathbf{K} :

$$\{\varinjlim \mathcal{N}_X; X \in \text{Ob}(\mathbf{K})\} \subseteq \Omega \subseteq \text{Mono}(\mathbf{K}),$$

(ii) the net \mathcal{N} generates the class Φ on the outside:

$$\mathcal{N} \subseteq \Phi \subseteq \mathcal{N} \circ \text{Mor}(\mathbf{K}).$$

Then

(a) for each object X in \mathbf{K} the local limit $\varinjlim \mathcal{N}_X$ is an imprint $\text{imp}_{\Phi}^{\Omega} X$ of the class Ω by means of the class Φ :

$$\varinjlim \mathcal{N}_X = \text{imp}_{\Phi}^{\Omega} X, \quad (1.91)$$

(b) if \mathbf{K} is injectively determined, then there is a covariant functor $(X, \alpha) \mapsto (\text{Imp}_{\Phi}^{\Omega} X, \text{imp}_{\Phi}^{\Omega} \alpha)$ from \mathbf{K} into \mathbf{K}

$$\text{imp}_{\Phi}^{\Omega}(1_X) = 1_{\text{Imp}_{\Phi}^{\Omega} X}, \quad \text{imp}_{\Phi}^{\Omega}(\beta \circ \alpha) = \text{imp}_{\Phi}^{\Omega} \beta \circ \text{imp}_{\Phi}^{\Omega} \alpha, \quad (1.92)$$

such that for any morphism $\alpha : X \rightarrow Y$ in \mathbf{K} the following diagram is commutative:

$$\begin{array}{ccc} X & \xleftarrow{\text{imp}_{\Phi}^{\Omega} X} & \text{Imp}_{\Phi}^{\Omega} X \\ \downarrow \alpha & & \downarrow \text{imp}_{\Phi}^{\Omega} \alpha \\ Y & \xleftarrow{\text{imp}_{\Phi}^{\Omega} Y} & \text{Imp}_{\Phi}^{\Omega} Y \end{array} . \quad (1.93)$$

• Let us say that in diagram (1.81)

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ \sigma \uparrow & & \uparrow \tau \\ \text{Dom}(\sigma) & \xrightarrow{\alpha_{\sigma}^{\tau}} & \text{Dom}(\tau) \end{array}$$

the shed τ of the element σ is *relatively splitted*, if for each morphism $\delta : Y \rightarrow \text{Ran}(\sigma)$ the commutativity of the diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ & \nwarrow \delta & \uparrow \tau \\ & & \text{Dom}(\tau) \end{array}$$

automatically implies the commutativity of the diagram

$$\begin{array}{ccc} X & & \\ \sigma \uparrow & \nwarrow \delta & \\ \text{Dom}(\sigma) & \xrightarrow{\alpha_{\sigma}^{\tau}} & \text{Dom}(\tau) \end{array}$$

If in a net of monomorphisms \mathcal{N} each element $\sigma \in \mathcal{N}_X$ has a relatively splitted shed for each morphism $\alpha : X \rightarrow Y$ in \mathbf{K} , then the net \mathcal{N} will be called a *relatively splitted net of monomorphisms*.

Theorem 1.18. *Suppose in a category \mathbf{K} we have a relatively splitted net of monomorphisms \mathcal{N} and classes of morphisms Ω and Φ such that:*

(i) *all the local limits $\varinjlim \mathcal{N}_X$ belong to Ω :*

$$\{\varinjlim \mathcal{N}_X; X \in \text{Ob}(\mathbf{K})\} \subseteq \Omega,$$

(ii) *the net \mathcal{N} generates the class Φ on the outside:*

$$\mathcal{N} \subseteq \Phi \subseteq \mathcal{N} \circ \text{Mor}(\mathbf{K}).$$

Then

(a) *for each object X in \mathbf{K} the local limit $\varinjlim \mathcal{N}_X$ is an imprint $\text{imp}_\Phi^\Omega X$ of the class Ω by means of the class Φ :*

$$\varinjlim \mathcal{N}_X = \text{imp}_\Phi^\Omega X, \quad (1.94)$$

(b) *if \mathbf{K} is injectively determined, then there is a covariant functor $(X, \alpha) \mapsto (\text{Imp}_\Phi^\Omega X, \text{imp}_\Phi^\Omega \alpha)$ from \mathbf{K} into \mathbf{K}*

$$\text{imp}_\Phi^\Omega(1_X) = 1_{\text{imp}_\Phi^\Omega X}, \quad \text{imp}_\Phi^\Omega(\beta \circ \alpha) = \text{imp}_\Phi^\Omega \beta \circ \text{imp}_\Phi^\Omega \alpha. \quad (1.95)$$

such tha for any morphism $\alpha : X \rightarrow Y$ in \mathbf{K} the following diagram is commutative

$$\begin{array}{ccc} X & \xleftarrow{\text{imp}_\Phi^\Omega X} & \text{Imp}_\Phi^\Omega X \\ \downarrow \alpha & & \downarrow \text{imp}_\Phi^\Omega \alpha \\ Y & \xleftarrow{\text{imp}_\Phi^\Omega Y} & \text{Imp}_\Phi^\Omega Y \end{array} . \quad (1.96)$$

§ 2 Nodal decomposition and its relations with envelopes and imprints

In this section we are going to discuss two special cases: the envelopes in the class **Epi** of all epimorphisms (with the dual construction, the imprints of the class **Mono** of all monomorphisms), and the envelopes in the class **SEpi** of all strong epimorphisms (with their duals, the imprints in the class **SMono** of all strong monomorphisms). These objects are curiously related to the notion of the nodal decomposition, which will be defined later on page 47.

(a) Nodal decomposition

Strong decompositions.

- A representation of a morphism φ into a composition of three morphisms

$$\varphi = \iota \circ \rho \circ \gamma,$$

where ι is a strong monomorphism, and γ a strong epimorphism, will be called a *strong decomposition* of φ .

Theorem 2.1. *If $\varphi = \iota \circ \rho \circ \gamma$ is a strong decomposition of φ , then for any other decomposition*

$$\varphi = \mu \circ \varepsilon$$

- *the epimorphity of ε implies the existence of a unique morphism μ' such that the following diagram is commutative:*

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \downarrow \gamma & \searrow \varepsilon & \nearrow \mu \\ & M & \\ & \searrow \mu' & \nearrow \iota \\ X' & \xrightarrow{\rho} & Y' \end{array} \quad (2.1)$$

(in this case if μ is a monomorphism, then μ' is a monomorphism as well),

- the monomorphy of μ implies the existence of a unique morphism ε' such that the following diagram is commutative:

$$\begin{array}{ccc}
 X & \xrightarrow{\varphi} & Y \\
 \downarrow \gamma & \searrow \varepsilon & \nearrow \mu \\
 & M & \\
 \downarrow \varepsilon' & \nearrow \rho & \downarrow \iota \\
 X' & \xrightarrow{\rho} & Y'
 \end{array} \quad (2.2)$$

(in this case if ε is an epimorphism, then ε' is an epimorphism as well).

Proof. Let ε be an epimorphism. Consider the diagram

$$\begin{array}{ccc}
 X & & Y \\
 \downarrow \gamma & \searrow \varepsilon & \nearrow \mu \\
 & M & \\
 \downarrow \rho & & \downarrow \iota \\
 X' & \xrightarrow{\rho} & Y'
 \end{array}$$

and transform it into the following one:

$$\begin{array}{ccc}
 X & \xrightarrow{\varepsilon} & M \\
 \downarrow \rho \circ \gamma & & \downarrow \mu \\
 Y' & \xrightarrow{\iota} & Y
 \end{array}$$

Here ε is an epimorphism, and μ a strong monomorphism, hence there exists a (unique) morphism μ' such that

$$\begin{array}{ccc}
 X & \xrightarrow{\varepsilon} & M \\
 \downarrow \rho \circ \gamma & \searrow \mu' & \downarrow \mu \\
 Y' & \xrightarrow{\iota} & Y
 \end{array}$$

This is the morphism for (2.1). By Property 1⁰ on page 7, if in addition $\mu = \iota \circ \mu'$ is a monomorphism, then μ' is also a monomorphism. The second case is dual. \square

Suppose we have two strong decompositions $\varphi = \iota \circ \rho \circ \gamma$ and $\varphi = \iota' \circ \rho' \circ \gamma'$ of one morphism φ

$$\begin{array}{ccc}
 X & \xrightarrow{\varphi} & Y \\
 \downarrow \gamma & & \downarrow \iota \\
 P & \xrightarrow{\rho} & Q
 \end{array}
 \quad
 \begin{array}{ccc}
 X & \xrightarrow{\varphi} & Y \\
 \downarrow \gamma' & & \downarrow \iota' \\
 P' & \xrightarrow{\rho'} & Q'
 \end{array}$$

If there exist (the only possible) morphisms $\sigma : P \rightarrow P'$ and $\tau : Q' \rightarrow Q$ such that the following diagram is commutative

$$\begin{array}{ccccc}
 X & & & & Y \\
 \downarrow \gamma & \searrow \gamma' & & \searrow \iota' & \downarrow \iota \\
 & P' & \xrightarrow{\rho'} & Q' & \\
 \downarrow \sigma & \nearrow \rho & & \nearrow \tau & \downarrow \iota \\
 P & & & & Q
 \end{array} \quad (2.3)$$

then we say that the strong decomposition $\varphi = \iota \circ \rho \circ \gamma$ is *subordinated* to the strong decomposition $\varphi = \iota' \circ \rho' \circ \gamma'$, and we write in this case

$$(\iota, \rho, \gamma) \leq (\iota', \rho', \gamma').$$

If in addition σ and τ are isomorphisms here, then we say that the decompositions $\varphi = \iota \circ \rho \circ \gamma$ and $\varphi = \iota' \circ \rho' \circ \gamma'$ are *isomorphic*, and we write

$$(\iota, \rho, \gamma) \cong (\iota', \rho', \gamma').$$

Proposition 2.1. *The two-sided inequality*

$$(\iota, \rho, \gamma) \leq (\iota', \rho', \gamma') \leq (\iota, \rho, \gamma)$$

is equivalent to the isomorphism of strong decompositions

$$(\iota, \rho, \gamma) \cong (\iota', \rho', \gamma').$$

Proof. The first inequality here implies the existence of (unique) dotted arrows in (2.3), and the second one means that the reverse arrows exist as well (and again are unique). In addition the epimorphy of γ and γ' imply that σ with its reverse arrow are mutually reverse isomorphisms, while the monomorphy of ι and ι' imply that the same is true for τ with its reverse arrow. \square

Nodal decomposition. If in a strong decomposition $\varphi = \iota' \circ \rho' \circ \gamma'$ the middle morphism ρ' is a bimorphism, then we call this a *nodal decomposition*. We say also that \mathbf{K} is a *category with a nodal decomposition*, if every morphism φ in \mathbf{K} has a nodal decomposition.

Proposition 2.2. *Each nodal decomposition $\varphi = \iota' \circ \rho' \circ \gamma'$ subordinates each strong decomposition $\varphi = \iota \circ \rho \circ \gamma$:*

$$(\iota, \rho, \gamma) \leq (\iota', \rho', \gamma').$$

As a corollary, a nodal decomposition is unique up to isomorphism.

Proof. Let $\varphi = \iota \circ \rho \circ \gamma$ be a strong decomposition. If we transform the diagram

$$\begin{array}{ccccc} X & & & & Y \\ & \searrow \gamma' & & \nearrow \iota' & \\ & P' & \xrightarrow{\rho'} & Q' & \\ & \downarrow \gamma & & & \downarrow \iota \\ P & \xrightarrow{\rho} & & & Q \end{array} \quad (2.4)$$

into the diagram

$$\begin{array}{ccc} X & & Y \\ & \searrow \rho' \circ \gamma' & \nearrow \iota' \\ & Q' & \\ & \searrow \rho \circ \gamma & \\ & & Q \end{array}$$

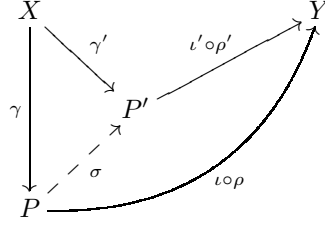
then one can recognize here a quadrangle of the form (0.16), since ι is a strong monomorphism here, and $\rho' \circ \gamma'$ an epimorphism (as a composition of an epimorphism γ' and a bimorphism ρ'). Hence, there is a unique morphism τ such that

$$\begin{array}{ccc} X & & Y \\ & \searrow \rho' \circ \gamma' & \nearrow \iota' \\ & Q' & \\ & \searrow \rho \circ \gamma & \nearrow \tau \\ & & Q \end{array}$$

Similarly, one can transform diagram (2.4) into

$$\begin{array}{ccc} X & & Y \\ & \searrow \gamma' & \nearrow \iota' \circ \rho' \\ & P' & \\ & \downarrow \gamma & \nearrow \iota \circ \rho \\ P & & Q \end{array}$$

and this again is a quadrangle of the form (0.16), since γ is a strong epimorphism here, and $\iota' \circ \rho'$ a monomorphism (as a composition of a bimorphism ρ' and a monomorphism ι'). Hence, there exists a unique morphism σ such that



These two morphisms together give diagram (2.3). \square

- From the uniqueness (up to isomorphism) of the nodal decomposition $\varphi = \iota' \circ \rho' \circ \gamma'$ it follows that one can assign notations to its components. We will further depict a nodal decomposition of a morphism $\varphi : X \rightarrow Y$ as a diagram

$$\begin{array}{ccc}
 X & \xrightarrow{\varphi} & Y \\
 \text{coim}_{\infty} \varphi \downarrow & & \uparrow \text{im}_{\infty} \varphi \\
 \text{Coim}_{\infty} \varphi & \xrightarrow{\text{red}_{\infty} \varphi} & \text{Im}_{\infty} \varphi
 \end{array} \tag{2.5}$$

(where elements are defined up to isomorphisms). The proof of Theorem 2.4 below and Remark 2.2 justify these notations, since they show that coim_{∞} , red_{∞} and im_{∞} can be conceived as a sort of “transfinite induction” of the usual operation coim , red and im in preabelian categories:

$$\begin{aligned}
 \text{coim}_{\infty} &= \lim_{n \rightarrow \infty} \underbrace{\text{coim} \circ \text{coim} \circ \dots \circ \text{coim}}_{n \text{ multipliers}} \\
 \text{red}_{\infty} &= \lim_{n \rightarrow \infty} \underbrace{\text{red} \circ \text{red} \circ \dots \circ \text{red}}_{n \text{ multipliers}} \\
 \text{im}_{\infty} &= \lim_{n \rightarrow \infty} \underbrace{\text{im} \circ \text{im} \circ \dots \circ \text{im}}_{n \text{ multipliers}}
 \end{aligned}$$

We will call

- $\text{im}_{\infty} \varphi$ a *nodal image*,
- $\text{red}_{\infty} \varphi$ a *nodal reduced part*,
- $\text{coim}_{\infty} \varphi$ a *nodal coimage*

of the morphism φ .

Remark 2.1. By Theorem 2.1,

- for any decomposition $\varphi = \mu \circ \varepsilon$, where ε is an epimorphism, there is a unique morphism μ' such that

$$\begin{array}{ccc}
 X & \xrightarrow{\varphi} & Y \\
 \varepsilon \searrow & \mu \nearrow & \\
 & M & \\
 \text{coim}_{\infty} \varphi \downarrow & & \uparrow \text{im}_{\infty} \varphi \\
 \text{Coim}_{\infty} \varphi & \xrightarrow{\text{red}_{\infty} \varphi} & \text{Im}_{\infty} \varphi
 \end{array} \tag{2.6}$$

(A dashed arrow labeled μ' goes from M to $\text{Im}_{\infty} \varphi$.)

(and if μ is a monomorphism, then μ' is a monomorphism),

- for any decomposition $\varphi = \mu \circ \varepsilon$, where μ is a monomorphism, there is a unique morphism ε' such that

$$\begin{array}{ccc}
 X & \xrightarrow{\varphi} & Y \\
 \varepsilon \searrow & \mu \nearrow & \\
 & M & \\
 \text{coim}_{\infty} \varphi \downarrow & & \uparrow \text{im}_{\infty} \varphi \\
 \text{Coim}_{\infty} \varphi & \xrightarrow{\text{red}_{\infty} \varphi} & \text{Im}_{\infty} \varphi
 \end{array} \tag{2.7}$$

(A dashed arrow labeled ε' goes from $\text{Coim}_{\infty} \varphi$ to M .)

(and if ε is an epimorphism, then ε' is an epimorphism).

Factorizations in a category with nodal decomposition. From (2.6) and (2.7) we immediately have

Proposition 2.3. *If $X \xrightarrow{\varepsilon} M \xrightarrow{\mu} Y$ is a factorization of a morphism $X \xrightarrow{\varphi} Y$ in a category \mathbf{K} with a nodal decomposition, then there are unique morphisms $\text{Coim}_\infty \varphi \xrightarrow{\varepsilon'} M$ and $M \xrightarrow{\mu'} \text{Im}_\infty \varphi$ such that the following diagram is commutative:*

$$\begin{array}{ccccc}
 X & \xrightarrow{\varphi} & Y & & \\
 \downarrow \text{coim}_\infty \varphi & \searrow \varepsilon & \nearrow \mu & & \uparrow \text{im}_\infty \varphi \\
 & M & & & \\
 \uparrow \varepsilon' & \nearrow \mu' & \downarrow & & \\
 \text{Coim}_\infty \varphi & \xrightarrow{\text{red}_\infty \varphi} & \text{Im}_\infty \varphi & &
 \end{array} \quad (2.8)$$

Moreover, ε' is an epimorphism, and μ' a monomorphism.

Let (ε, μ) and (ε', μ') be two factorizations of φ . We say that the factorization (ε, μ) is *subordinated* to the factorization (ε', μ') (or (ε', μ') *subordinates* (ε, μ)), and write

$$(\varepsilon, \mu) \leq (\varepsilon', \mu'),$$

if there exists a morphism β such that

$$\varepsilon' = \beta \circ \varepsilon, \quad \mu = \mu' \circ \beta$$

i.e.

$$\begin{array}{ccc}
 X & \xrightarrow{\varphi} & Y \\
 \downarrow \varepsilon & \searrow & \nearrow \mu' \\
 M & \xrightarrow{\beta} & M' \\
 & \nearrow \varepsilon' & \downarrow \mu
 \end{array}$$

From Properties 1° and 3° on page 7 it follows that β , if exists, must be a bimorphism, and from the fact that μ' is a monomorphism (or from the fact that ε is an epimorphism) that β is unique.

Theorem 2.2. *In a category \mathbf{K} with nodal decomposition*

- (i) *every morphism φ has a factorization,*
- (ii) *among all factorizations of φ there is a minimal $(\varepsilon_{\min}, \mu_{\min})$ and a maximal $(\varepsilon_{\max}, \mu_{\max})$, i.e. the factorizations that bound any other factorization (ε, μ) :*

$$(\varepsilon_{\min}, \mu_{\min}) \leq (\varepsilon, \mu) \leq (\varepsilon_{\max}, \mu_{\max})$$

Proof. Here (i) follows from (ii), so we prove (ii). Put

$$\varepsilon_{\min} = \text{coim}_\infty \varphi, \quad \mu_{\min} = \text{im}_\infty \varphi \circ \text{red}_\infty \varphi, \quad \varepsilon_{\max} = \text{red}_\infty \varphi \circ \text{coim}_\infty \varphi, \quad \mu_{\max} = \text{im}_\infty \varphi$$

then these will be factorizations of φ , and from (2.8) it follows that the first one of them is minimal, and the second one is maximal. \square

Strong morphisms in a category with nodal decomposition.

Theorem 2.3. *In a category with nodal decomposition*

- (a) μ is an immediate monomorphism $\iff \mu$ is a strong monomorphism $\iff \mu \cong \text{im}_\infty \mu \iff \text{coim}_\infty \mu$ and $\text{red}_\infty \mu$ are isomorphisms,
- (b) ε is an immediate epimorphism $\iff \varepsilon$ is a strong epimorphism $\iff \varepsilon \cong \text{coim}_\infty \varepsilon \iff \text{im}_\infty \mu$ and $\text{red}_\infty \mu$ are isomorphisms.

Proof. By the duality principle it is sufficient to prove (a).

1. If $\mu : X \rightarrow Y$ is an immediate monomorphism, then in its maximal factorization

$$\mu = \mu_{\max} \circ \varepsilon_{\max}$$

the morphism $\varepsilon_{\max} = \text{red}_{\infty} \mu \circ \text{coim}_{\infty} \mu$ must be an isomorphism. This implies formula

$$1_X = (\varepsilon_{\max})^{-1} \circ \text{red}_{\infty} \mu \circ \text{coim}_{\infty} \mu$$

from which one can conclude that $\text{coim}_{\infty} \mu$ is a coretraction. On the other hand, $\text{coim}_{\infty} \mu$ is an epimorphism, hence $\text{coim}_{\infty} \mu$ is an isomorphism. This implies that $\text{red}_{\infty} \mu = \varepsilon_{\max} \circ (\text{coim}_{\infty} \mu)^{-1}$ is an isomorphism.

2. If $\text{coim}_{\infty} \mu$ and $\text{red}_{\infty} \mu$ are isomorphisms, then its composition $\chi = \text{red}_{\infty} \mu \circ \text{coim}_{\infty} \mu$ is an isomorphism as well, and at the same time $\mu = \text{im}_{\infty} \mu \circ \chi$. This means that $\mu \cong \text{im}_{\infty} \mu$.

3. If $\mu \cong \text{im}_{\infty} \mu$, then, since $\text{im}_{\infty} \mu$ is a strong monomorphism, μ is also a strong monomorphism.

4. If μ is a strong monomorphism, then by property 2° on page 14, μ is an immediate monomorphism. \square

On existence of a nodal decomposition. Let us note that if μ is a monomorphism in a category K , then for any its decomposition $\mu = \mu' \circ \varepsilon$, if ε is a strong epimorphism, then ε must be an isomorphism:

$$\mu \in \text{Mono} \implies \left(\forall \varepsilon \in \text{SEpi} \quad \forall \mu' \quad \mu = \mu' \circ \varepsilon \implies \varepsilon \in \text{Iso} \right).$$

Indeed, by 1° on p.7, the equality $\mu = \mu' \circ \varepsilon$ means that ε must be an isomorphism, and, since in addition ε is a strong epimorphism, so (by 4° on p.14), an immediate epimorphism, then by 4° on p.13 we obtain that ε is an isomorphism.

- Let us say that in a category K *strong epimorphisms discern monomorphisms*, if the reverse is true: from the fact that a morphism μ is not a monomorphism it follows that μ can be represented as a composition $\mu = \mu' \circ \varepsilon$, where ε is a strong epimorphism, which is not an isomorphism.

Dually, if ε is an epimorphism in a category K , then for any its decomposition $\varepsilon = \mu \circ \varepsilon'$, if μ is a strong monomorphism, then μ must be an isomorphism:

$$\varepsilon \in \text{Epi} \implies \left(\forall \mu \in \text{SMono} \quad \forall \varepsilon' \quad \varepsilon = \mu \circ \varepsilon' \implies \mu \in \text{Iso} \right).$$

- Let us say that in a category K *strong monomorphisms discern epimorphisms*, if the reverse is true: from the fact that a morphism ε is not an epimorphism it follows that ε can be represented as a composition $\varepsilon = \mu \circ \varepsilon'$, where μ is a strong monomorphism, which is not an isomorphism.

Recall that the notion of linearly complete category was introduced on page 3.

Theorem 2.4. *Let K be a linearly complete, strongly well-powered and strongly co-well-powered category, where strong epimorphisms discern monomorphisms, and, dually, strong monomorphisms discern epimorphisms. Then K is a category with nodal decomposition.*

Before proving this theorem let us introduce the following auxiliary construction. Take a morphism $\varphi : X \rightarrow Y$ in a category K . Since K is strongly co-well-powered, in the category $\text{SEpi}(X)$ of strong epimorphisms going from X there exists a set of strong quotient objects $Q \subseteq \text{SEpi}(X)$, and in the category $\text{SMono}(Y)$ of strong monomorphisms coming to Y there is a set of strong subobjects $S \subseteq \text{SMono}(Y)$. We freeze these sets Q and S .

- A decomposition $\varphi = \iota \circ \rho \circ \gamma$ of a morphism φ is said to be *admissible*, if $\gamma \in Q$ and $\iota \in S$. Certainly, any strong decomposition $\varphi = \iota' \circ \rho' \circ \gamma'$ of a morphism φ is isomorphic to some admissible decomposition $\varphi = \iota \circ \rho \circ \gamma$.
- Let us call a *local basic decomposition* of a morphism φ in a category K an arbitrary map $\rho \mapsto (\text{coim } \rho, \text{red } \rho, \text{im } \rho)$ that to each admissible decomposition (ι, ρ, γ) of the morphism φ assigns some strong decomposition $(\text{im } \rho, \text{red } \rho, \text{coim } \rho)$ of ρ

$$\begin{array}{ccc}
 X & \xrightarrow{\varphi} & Y \\
 \gamma \downarrow & & \uparrow \iota \\
 \text{Dom } \rho & \xrightarrow{\rho} & \text{Ran } \rho \\
 \text{coim } \rho \downarrow & & \uparrow \text{im } \rho \\
 \text{Coim } \rho & \xrightarrow{\text{red } \rho} & \text{Im } \rho
 \end{array} \tag{2.9}$$

in such a way that the following conditions are fulfilled:

- (a) the decomposition $(\iota \circ \text{im } \rho, \text{red } \rho, \text{coim } \rho \circ \gamma)$ of φ is admissible (i.e. $\text{coim } \rho \circ \gamma \in Q$ and $\iota \circ \text{im } \rho \in S$),
- (b) ρ is a monomorphism $\iff \text{coim } \rho$ is an isomorphism $\iff \text{coim } \rho = 1$,
- (c) ρ is an epimorphism $\iff \text{im } \rho$ is an epimorphism $\iff \text{im } \rho = 1$.

Lemma 2.1. *Let \mathbf{K} be a strongly well-powered and strongly co-well-powered category, where strong epimorphisms discern monomorphisms, and strong monomorphisms discern epimorphisms. Then each morphism φ in \mathbf{K} has local basic decomposition.*

Proof. Let us first show that for any admissible decomposition (ι, ρ, γ) of φ a diagram (2.9) satisfying (a), (b), (c) exists. Let us freeze this decomposition (ι, ρ, γ) and consider several cases.

1. If ρ is not a monomorphism, then there exists a decomposition $\rho = \rho' \circ \varepsilon$, where ε is a strong epimorphism, but not an isomorphism. Set $\text{coim } \rho = \varepsilon$ and consider the morphism ρ' .
 - 1.1. If ρ' is not an epimorphism, then there exists a decomposition $\rho' = \mu \circ \rho''$, where μ is a strong monomorphism, but not an isomorphism. Then we set $\text{im } \rho = \mu$ and $\text{red } \rho = \rho''$.
 - 1.2. If ρ' is an epimorphism, then we set $\text{im } \rho = 1_{\text{Ran } \rho}$ and $\text{red } \rho = \rho'$.
2. If ρ is a monomorphism, then we set $\text{coim } \rho = 1_{\text{Dom } \rho}$ and again consider ρ .
 - 2.1. If ρ is not an epimorphism, then there exists a decomposition $\rho = \mu \circ \rho'$, where μ is a strong monomorphism, but not an isomorphism. We set $\text{im } \rho = \mu$ and $\text{red } \rho = \rho'$.
 - 2.2. If ρ is an epimorphism, then we set $\text{im } \rho = 1_Y$ and $\text{red } \rho = \rho$.

In any case we obtain a decomposition $\rho = \text{im } \rho \circ \text{red } \rho \circ \text{coim } \rho$, where $\text{im } \rho$ is a strong monomorphism, $\text{coim } \rho$ is a strong epimorphism, and (b) and (c) are fulfilled. Now to provide (a) we have to replace (if necessary) the epimorphism $\text{coim } \rho$ with an isomorphic epimorphism $\pi \circ \text{coim } \rho$ in such a way that $\pi \circ \text{coim } \rho \circ \gamma \in Q$, and this can be done due to Proposition 0.11. Similarly, the monomorphism $\text{im } \rho$ should be replaced with an isomorphic monomorphism $\text{im } \rho \circ \sigma$ in such a way that $\iota \circ \text{im } \rho \circ \sigma \in S$, and this can be done due to Proposition 0.9.

Thus, for an arbitrary admissible decomposition (ι, ρ, γ) of φ diagram (2.9) satisfying (a), (b), (c), exists. Note now that from Propositions 0.4 and 0.4 it follows that for a given admissible decomposition (ι, ρ, γ) of morphism φ the class of decompositions $(\text{im } \rho, \text{red } \rho, \text{coim } \rho)$ of ρ , which satisfy (a), (b), (c), is a set. Indeed, every such a decomposition $(\text{im } \rho, \text{red } \rho, \text{coim } \rho)$ is uniquely defined by the morphisms $\text{im } \rho$ and $\text{coim } \rho$ (since from monomorphicity of $\text{im } \rho$ and epimorphicity of $\text{coim } \rho$ it follows that $\text{red } \rho$, if exists, is unique). So the class of decompositions $(\text{im } \rho, \text{red } \rho, \text{coim } \rho)$ can be conceived as a subclass in the cartesian product of sets $A \times B$, where $A = \{\alpha \in \text{Sub}(\text{Ran } \rho) : \iota \circ \alpha \in S\}$ is a class of monomorphisms where $\text{im } \rho$ runs, and which is a set by Proposition 0.9, and $B = \{\beta \in \text{Quot}(E) : \beta \circ \varepsilon \in Q\}$ is a class of epimorphisms, where $\text{coim } \rho$ runs, and which is a set by Proposition 0.11).

We obtain that for any admissible decomposition (ι, ρ, γ) of φ the class of decompositions $(\text{coim } \rho, \text{red } \rho, \text{im } \rho)$ satisfying (2.9) and (a), (b), (c), is a (non-empty) set. From this it follows that we can apply the axiom of choice and construct a map which to each admissible decomposition (ι, ρ, γ) of φ assigns a decomposition $(\text{coim } \rho, \text{red } \rho, \text{im } \rho)$, satisfying (2.9) and (a), (b), (c). This is the required map $\rho \mapsto (\text{coim } \rho, \text{red } \rho, \text{im } \rho)$. \square

Proof of Theorem 2.4. Take a morphism $\varphi : X \rightarrow Y$, find a set of strong quotient objects $Q \subseteq \text{SEpi}(X)$ and a set of strong subobjects $S \subseteq \text{SMono}(Y)$, and construct a local basic decomposition like in Lemma 2.1. The proof consists in constructing a transfinite system of objects and morphisms, indexed by ordinal numbers $i \in \mathbf{Ord}$,

$$X^i \xrightarrow{\varphi^i} Y^i, \quad X^i \xrightarrow{\varepsilon_j^i} X^j, \quad Y^i \xleftarrow{\mu_j^i} Y^j \quad (i \leq j)$$

the idea of which is illustrated by the following diagram (going infinitely below):

$$\begin{array}{ccc}
 X & \xrightarrow{\varphi} & Y \\
 \parallel 1_X & & \parallel 1_Y \\
 X^0 & \xrightarrow{\varphi^0 = \varphi} & Y^0 \\
 \downarrow \varepsilon_1^0 = \text{coim } \varphi^0 & & \uparrow \mu_1^0 = \text{im } \varphi^0 \\
 X^1 & \xrightarrow{\varphi^1 = \text{red } \varphi^0} & Y^1 \\
 \downarrow \varepsilon_2^1 = \text{coim } \varphi^1 & & \uparrow \mu_2^1 = \text{im } \varphi^1 \\
 X^2 & \xrightarrow{\varphi^2 = \text{red } \varphi^1} & Y^1 \\
 \downarrow \varepsilon_3^2 = \text{coim } \varphi^2 & & \uparrow \mu_3^2 = \text{im } \varphi^2 \\
 \vdots & & \vdots
 \end{array} \tag{2.10}$$

Here is how we do this.

0) Initially, we put

$$X^0 = X, \quad Y^0 = Y, \quad \varphi^0 = \varphi, \quad \varepsilon_1^0 = \text{coim } \varphi^0, \quad \mu_1^0 = \text{im } \varphi^0, \quad \varphi^1 = \text{red } \varphi^0.$$

1) Then for an arbitrary ordinal number k we put

$$\varepsilon_k^k = 1_{X^k}, \quad \mu_k^k = 1_{Y^k}$$

and

— if k is an isolated ordinal, i.e. $k = j + 1$ for some j , then we set

$$\begin{aligned}
 X^k &= X^{j+1} = \text{Coim } \varphi^j, & Y^k &= Y^{j+1} = \text{Im } \varphi^j, \\
 \varepsilon_k^j &= \varepsilon_{j+1}^j = \text{coim } \varphi^j, & \mu_k^j &= \mu_{j+1}^j = \text{im } \varphi^j, & \varphi^k &= \varphi^{j+1} = \text{red } \varphi^j
 \end{aligned}$$

and after that for any other ordinal number $i < j$

$$\varepsilon_k^i = \varepsilon_{j+1}^i = \varepsilon_{j+1}^j \circ \varepsilon_j^i, \quad \mu_k^i = \mu_{j+1}^i = \mu_j^i \circ \mu_{j+1}^j,$$

— if k is a limit ordinal, i.e. for any $j < k$ we have $j + 1 < k$, then X^k is defined as the injective limit of the covariant system $\{X^j, \varepsilon_j^i; i \leq j < k\}$, Y^k as the projective limit of the contravariant system $\{Y^j, \mu_j^i; i \leq j < k\}$,

$$X^k = \lim_{j \rightarrow k} X^j, \quad Y^k = \lim_{k \leftarrow j} Y^j,$$

the system of morphisms $\{\varepsilon_k^i; i < k\}$ is the corresponding injective cone of morphism going to X^k , and the system of morphisms $\{\mu_k^i; i < k\}$ is the corresponding projective cone of morphisms going from Y^k ,

$$\varepsilon_k^i = \lim_{j \rightarrow k} \varepsilon_j^i, \quad \mu_k^i = \lim_{k \leftarrow j} \mu_j^i, \quad i \leq k.$$

This automatically implies equalities

$$\varepsilon_k^i = \varepsilon_k^j \circ \varepsilon_j^i, \quad \mu_k^i = \mu_j^i \circ \mu_k^j, \quad i \leq j \leq k$$

and by Proposition 0.13 all the morphisms ε_k^i are strong epimorphisms, while by Proposition 0.12 all the morphisms μ_j^i are strong monomorphisms. As a corollary, the object X^k can be chosen in such a way that the epimorphism ε_k^0 lies in Q (for this we just need to multiply from the left the system $\{\varepsilon_k^i; i < k\}$ of epimorphisms by a morphism, so that the property of being injective cone is preserved); similarly, the object Y^k can be chosen in such a way that the monomorphism μ_k^0 lies in the set S (for this we just need to multiply from the right the system $\{\mu_k^i; i < k\}$ of monomorphisms, so that the property of being projective cone is preserved). That is what we will do, and after that the morphism φ^k can be defined by two equivalent formulas:

$$\varphi^k = \lim_{k \leftarrow i} \lim_{j \rightarrow k} \mu_j^i \circ \varphi^j = \lim_{i \rightarrow k} \lim_{k \leftarrow j} \varphi^j \circ \varepsilon_j^i$$

Here the first double limit should be understood as follows: for a given $i < k$ the family $\{\mu_j^i \circ \varphi^j; i \leq j < k\}$ is an injective cone of the covariant system $\{\varepsilon_j^l; i \leq l, j < k\}$, so there exists a limit

$$\lim_{j \rightarrow k} \mu_j^i \circ \varphi^j;$$

after that the system $\{\lim_{j \rightarrow k} \mu_j^i \circ \varphi^j; i < k\}$ turns out to be a projective cone of the contravariant system $\{\mu_j^l; i \leq l, j < k\}$, so there exists a limit

$$\lim_{k \leftarrow i} \lim_{j \rightarrow k} \mu_j^i \circ \varphi^j.$$

Similarly, in the second double limit for a given $i < k$ the family $\{\varphi^j \circ \varepsilon_j^i; i \leq j < k\}$ is a projective cone of the contravariant system $\{\mu_j^l; i \leq l, j < k\}$, so there exists a limit

$$\lim_{k \leftarrow j} \varphi^j \circ \varepsilon_j^i;$$

after that the system $\{\lim_{k \leftarrow j} \varphi^j \circ \varepsilon_j^i; i < k\}$ turns out to be an injective cone of the covariant system $\{\varepsilon_j^l; i \leq l, j < k\}$, so there exists a limit

$$\lim_{i \rightarrow k} \lim_{k \leftarrow j} \varphi^j \circ \varepsilon_j^i.$$

Each of these double limits gives an arrow from X^k into Y^k which makes all the necessary diagrams commutative, and since this arrow is unique (this follows from the fact that μ_k^i are monomorphisms and ε_k^i are epimorphisms), those double limits (arrows) coincide.

Eventually we obtain a system of morphisms such that for any two ordinal numbers $i \leq j$ the following diagram is commutative

$$\begin{array}{ccc} X^i & \xrightarrow{\varphi^i} & Y^i \\ \varepsilon_j^i \downarrow & & \uparrow \mu_j^i \\ X^j & \xrightarrow{\varphi^j} & Y^j \end{array}$$

and for any three ordinal numbers $i \leq j \leq k$ the following diagrams are commutative

$$\begin{array}{ccc} & X^i & \\ \varepsilon_j^i \swarrow & \downarrow \varepsilon_k^i & \searrow \varepsilon_k^j \\ X^j & & X^k \end{array} \quad \begin{array}{ccc} & Y^i & \\ \mu_j^i \swarrow & \downarrow \mu_k^i & \searrow \mu_k^j \\ Y^j & & Y^k \end{array}$$

and moreover, ε_j^i are strong epimorphisms, and μ_j^i are strong monomorphisms. From the last two diagrams it follows that the formulas

$$\begin{cases} F(i) = \varepsilon_i^0, & i \in \mathbf{Ord} \\ F(i, j) = \varepsilon_j^i, & i \leq j \in \mathbf{Ord} \end{cases} \quad \begin{cases} G(i) = \mu_i^0, & i \in \mathbf{Ord} \\ G(i, j) = \mu_j^i, & i \leq j \in \mathbf{Ord} \end{cases}$$

define a covariant functor $F : \mathbf{Ord} \rightarrow Q$ and a contravariant functor $G : \mathbf{Ord} \rightarrow S$. Since Q and S are sets, by Theorem 0.1 these functors must stabilize, i.e. starting from some ordinal number k (which can be chosen common for F and G) the morphisms $F(i, j)$ and $G(i, j)$ become isomorphisms. Since in addition the categories Q and S are partially ordered classes (and as a corollary, only local identities are isomorphisms there, by Proposition 0.1), we obtain (following Remark 0.1) that diagram (2.10) is stabilized in the sense that, starting from some k ,

— the objects X^l become the same, and the morphisms ε_m^l become local identities of X^k :

$$\forall m > l \geq k \quad X^m = X^l = X^k, \quad \varepsilon_m^l = 1_{X^k}$$

— and the objects Y^l become the same and the morphisms μ_m^l become local identities of Y^k :

$$\forall m > l \geq k \quad Y^m = Y^l = Y^k, \quad \mu_m^l = 1_{Y^k}$$

Now let us consider the diagram

$$\begin{array}{ccc}
 X & \xrightarrow{\varphi} & Y \\
 \varepsilon_k^0 \downarrow & & \uparrow \mu_k^0 \\
 X^k & \xrightarrow{\varphi^k} & Y^k
 \end{array} \tag{2.11}$$

Here ε_k^0 is a strong epimorphism, and μ_k^0 a strong monomorphism. From the equality $\varepsilon_{k+1}^k = \text{coim } \varphi^k = 1_{X^k}$ (which holds since the sequence ε_j^0 is stabilized for $j \geq k$) it follows by condition (b) on page 51, that φ^k is a monomorphism. On the other hand, from the equality $\mu_{k+1}^k = \text{im } \varphi^k = 1_{Y^k}$ (which holds since the sequence μ_j^0 is stabilized for $j \geq k$) it follows by condition (c) on page 51, that φ^k is an epimorphism. Thus, φ^k is a bimorphism, hence (2.11) is a nodal decomposition for φ . \square

Connection with the basic decomposition in pre-Abelian categories Let us discuss the obvious analogy between nodal decomposition and the decomposition of a morphism φ in a pre-Abelian category \mathbf{K} into a coimage $\text{coim } \varphi$, image $\text{im } \varphi$ and a morphism between them which we denote by $\text{red } \varphi$.

Recall (see definition in [8] or in [11]) that *pre-Abelian category* is an enriched category \mathbf{K} over the category \mathbf{Ab} of Abelian groups, which is finitely complete and has zero object. In such a category every morphism $\varphi : X \rightarrow Y$ has a kernel and a cokernel. From this it follows that φ can be represented as a composition

$$\begin{array}{ccc}
 X & \xrightarrow{\varphi} & Y \\
 \text{coim } \varphi \downarrow & & \uparrow \text{im } \varphi \\
 \text{Coim } \varphi & \xrightarrow{\text{red } \varphi} & \text{Im } \varphi
 \end{array} \tag{2.12}$$

where the morphism $\text{coim } \varphi = \text{coker}(\ker \varphi)$ is called the *coimage* of φ , the morphism $\text{im } \varphi = \ker(\text{coker } \varphi)$ the *image* of φ , and the existence and uniqueness of the morphism $\text{red } \varphi$ is proved separately, and we will call it the *reduced part* of φ .

- The representation of a morphism φ as a composition (2.12) we call the *basic decomposition* of φ .

It is known (see [5, Proposition 4.3.6(4)]) that in a pre-Abelian category (in fact, in a category with zero) every kernel $\ker \varphi$ (and thus, every image $\text{im } \varphi$) is always a strong monomorphism, and every cokernel $\text{coker } \varphi$ (and thus, every coimage $\text{coim } \varphi$) is a strong epimorphism. As a corollary, we have

Theorem 2.5. *In a pre-Abelian category every basic decomposition is strong.*

This implies that *if a category \mathbf{K} is Abelian, then every basic decomposition in \mathbf{K} is nodal*. But if \mathbf{K} is not Abelian, then these decompositions do not necessarily coincide, see below Example 3.10.

The following two propositions are obvious:

Proposition 2.4. *In a pre-Abelian category for a morphism $\varphi : X \rightarrow Y$ the following conditions are equivalent:*

- (i) φ is a monomorphism,
- (ii) the zero morphism $0_{0,X}$ is a kernel for φ : $0_{0,X} = \ker \varphi$,
- (iii) the identity morphism 1_X is a cokernel for φ : $1_X = \text{coim } \varphi$.
- (iv) $\text{coim } \varphi$ is an isomorphism.

Proposition 2.5. *In a pre-Abelian category for a morphism $\varphi : X \rightarrow Y$ the following conditions are equivalent:*

- (i) φ is an epimorphism,
- (ii) the zero morphism $0_{Y,0}$ is a cokernel for φ : $0_{Y,0} = \text{coker } \varphi$,
- (iii) the identity morphism 1_Y is an image for φ : $1_Y = \text{im } \varphi$,
- (iv) $\text{im } \varphi$ is an isomorphism.

They imply

Proposition 2.6. *In a pre-Abelian category \mathbf{K} the strong epimorphisms discern monomorphisms and the strong monomorphisms discern epimorphisms.*

Proof. Consider the basic decomposition of $\varphi : X \rightarrow Y$:

$$\varphi = \text{im } \varphi \circ \text{red } \varphi \circ \text{coim } \varphi$$

If $\varphi : X \rightarrow Y$ is not a monomorphism, then by Proposition 2.4, $\text{coim } \varphi$ is not an isomorphism. On the other hand, by Proposition 2.5, $\text{coim } \varphi$ is a strong epimorphism. So, if we set $\varphi' = \text{im } \varphi \circ \text{red } \varphi$, then in the decomposition $\varphi = \varphi' \circ \text{coim } \varphi$ the morphism $\text{coim } \varphi$ is a strong epimorphism, but not an isomorphism. This means that strong epimorphisms discern monomorphisms in \mathbf{K} . The statement about strong monomorphisms is proved similarly. \square

Proposition 2.6 implies that if a pre-Abelian category \mathbf{K} is strongly well-powered and strongly co-well-powered, then \mathbf{K} has local basic decomposition (defined on page 50): the map $(\iota, \rho, \gamma) \mapsto (\text{coim } \rho, \text{red } \rho \text{ im } \rho)$ that to each admissible decomposition (ι, ρ, γ) (admissible decompositions were defined on page 50) of a given morphism φ assigns the basic decomposition of ρ , is a local basic decomposition of φ . Hence, the sufficient condition for existence of nodal decomposition (Theorem 2.4) becomes more simple:

Theorem 2.6. *If a pre-Abelian category \mathbf{K} is strongly well-powered and strongly co-well-powered, then every morphism $\varphi : X \rightarrow Y$ in \mathbf{K} has nodal decomposition (2.5).*

Remark 2.2. From Proposition 2.6 and Diagram (2.10) it follows that

- the nodal reduced part $\text{red}_\infty \varphi$ in diagram (2.5) can be conceived as a “limit” of transfinite sequence of “usual” reduced morphisms $\varphi^{i+1} = \text{red } \varphi^i$,
- the nodal coimage $\text{coim}_\infty \varphi$ is an injective limit of transfinite sequence of “usual” coimages $\text{coim } \varphi^i$ of this system of morphisms, and
- the nodal image $\text{im}_\infty \varphi$ is a projective limit of transfinite sequence of “usual” images $\text{im } \varphi^i$ of this system of morphisms.

Remark 2.3. Since as we already noticed the basic decomposition $\varphi = \text{im } \varphi \circ \text{red } \varphi \circ \text{coim } \varphi$ is strong, and thus, by Proposition 2.2, is subordinated to the nodal decomposition, there must exist unique morphisms σ and τ such that the following diagram is commutative:

$$\begin{array}{ccccc}
 X & \xrightarrow{\varphi} & Y & & \\
 \text{coim } \varphi \downarrow & \searrow \text{coim}_\infty \varphi & \nearrow \text{im}_\infty \varphi & & \downarrow \text{im } \varphi \\
 & \text{Coim}_\infty \varphi & \xrightarrow{\text{red}_\infty \varphi} & \text{Im}_\infty \varphi & \\
 \text{Coim } \varphi \swarrow \sigma & & & & \searrow \tau \\
 & \text{Coim } \varphi & \xrightarrow{\text{red } \varphi} & \text{Im } \varphi &
 \end{array} \tag{2.13}$$

At the same time, by Theorem 2.1,

- for any decomposition $\varphi = \mu \circ \varepsilon$, where ε is an epimorphism, there exists a unique morphism μ' such that the following diagram is commutative:

$$\begin{array}{ccccc}
 X & \xrightarrow{\varphi} & Y & & \\
 \text{coim } \varphi \downarrow & \searrow \varepsilon & \nearrow \mu & & \downarrow \text{im } \varphi \\
 & M & \xrightarrow{\text{im}_\infty \varphi} & \text{Im}_\infty \varphi & \\
 \text{Coim } \varphi \swarrow \sigma & \searrow \mu' & & & \searrow \tau \\
 & \text{Coim}_\infty \varphi & \xrightarrow{\text{red}_\infty \varphi} & \text{Im}_\infty \varphi & \\
 \text{Coim } \varphi \swarrow \sigma & & & & \searrow \tau \\
 & \text{Coim } \varphi & \xrightarrow{\text{red } \varphi} & \text{Im } \varphi &
 \end{array} \tag{2.14}$$

(in addition, if μ is a monomorphism, then μ' is a monomorphism as well);

- for any decomposition $\varphi = \mu \circ \varepsilon$, where μ is a monomorphism, there exists a unique morphism ε' such that the following diagram is commutative:

$$\begin{array}{ccccc}
 X & & & & Y \\
 & \searrow \varepsilon & & \nearrow \mu & \\
 & & M & & \\
 \text{coim } \varphi \downarrow & \searrow \text{coim}_\infty \varphi & \nearrow \varepsilon' & \nearrow \text{im}_\infty \varphi & \downarrow \text{im } \varphi \\
 & & \text{Coim}_\infty \varphi & \xrightarrow{\text{red}_\infty \varphi} & \text{Im}_\infty \varphi \\
 \sigma \nearrow & & & & \searrow \tau \\
 \text{Coim } \varphi & \xrightarrow{\text{red } \varphi} & & & \text{Im } \varphi
 \end{array} \tag{2.15}$$

(in addition, if ε is an epimorphism, then ε' is an epimorphism as well);

- in particular, for any factorization $\varphi = \mu \circ \varepsilon$ of φ there exist unique morphisms $\text{Coim } \varphi \xrightarrow{\varepsilon'} M$ and $M \xrightarrow{\mu'} \text{Im } \varphi$ such that the following diagram is commutative:

$$\begin{array}{ccccc}
 X & & & & Y \\
 & \searrow \varepsilon & & \nearrow \mu & \\
 & & M & & \\
 \text{coim } \varphi \downarrow & \searrow \text{coim}_\infty \varphi & \nearrow \varepsilon' & \nearrow \mu' & \downarrow \text{im } \varphi \\
 & & \text{Coim}_\infty \varphi & \xrightarrow{\text{red}_\infty \varphi} & \text{Im}_\infty \varphi \\
 \sigma \nearrow & & & & \searrow \tau \\
 \text{Coim } \varphi & \xrightarrow{\text{red } \varphi} & & & \text{Im } \varphi
 \end{array} \tag{2.16}$$

and in addition, ε' is an epimorphism, and μ' a monomorphism.

(b) Connections with envelopes in Epi and with imprints of Mono

Nodal decomposition in a category with envelopes in Epi and imprints of Mono. By analogy with definitions on p.50 we will say that in a category \mathbf{K}

- *epimorphisms discern monomorphisms*, if from the fact that a morphism μ is not a monomorphism it follows that μ can be represented as a composition $\mu = \mu' \circ \varepsilon$, where ε is an epimorphism, which is not an isomorphism,
- *monomorphisms discern epimorphisms*, if from the fact that a morphism ε is not an epimorphism it follows that ε can be represented as a composition $\varepsilon = \mu \circ \varepsilon'$, where μ is a monomorphism, which is not an isomorphism.

Theorem 2.7. *Suppose that in a category \mathbf{K}*

- epimorphisms discern monomorphisms, and, dually, monomorphisms discern epimorphisms,*
- every immediate monomorphism is a strong monomorphism, and, dually, every immediate epimorphism is a strong epimorphism,*
- every object X has an envelope in the category \mathbf{K} with respect to any morphism, starting from X , and, dually, in every object X there is an imprint of the category \mathbf{K} by means of any morphism coming to X .*

Then \mathbf{K} is a category with nodal decomposition.

Proof. Consider a morphism $\varphi : X \rightarrow Y$.

- Suppose $\varepsilon : X \rightarrow N$ is an envelope of X with respect to φ , and denote by β the dashed arrow in (1.5):

$$\varphi = \beta \circ \varepsilon$$

Note first that β is a monomorphism. Indeed, if β is not a monomorphism, then by (a), there exists a decomposition $\beta = \beta' \circ \pi$, where π is an epimorphism, but not an isomorphism. If we denote by N' the range of π , then we get a diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \varepsilon \downarrow & \searrow \beta & \uparrow \beta' \\ N & \xrightarrow{\pi} & N' \end{array} \quad (2.17)$$

where by definition $\varepsilon' = \pi \circ \varepsilon$, and this will be is epimorphism, as a composition of two epimorphisms. Thus, ε' is another extension of X with respect to φ . Hence, there exists a unique morphism v such that the following diagram is commutative:

$$\begin{array}{ccc} & X & \\ \varepsilon \swarrow & & \searrow \varepsilon' \\ N & \xleftarrow{v} & N' \end{array}$$

Here we have:

$$\pi \circ \varepsilon = \varepsilon' \implies v \circ \pi \circ \varepsilon = v \circ \varepsilon' = \varepsilon = 1_N \circ \varepsilon \implies v \circ \pi = 1_N$$

and

$$v \circ \varepsilon' = \varepsilon \implies \pi \circ v \circ \varepsilon' = \pi \circ \varepsilon = \varepsilon' = 1_{N'} \circ \varepsilon' \implies \pi \circ v = 1_{N'}.$$

I.e. π must be an isomorphism, and this contradicts our assumption that π is not an isomorphism.

2. Similarly one can prove that β is an immediate monomorphism. Indeed, any its factorization $\beta = \beta' \circ \pi$ leads again to diagram (2.17), and the same reasoning gives that π is an isomorphism.

3. The fact that β is an immediate monomorphism together with condition (b) imply that β is a strong monomorphism.

4. Denote by $\mu : M \rightarrow Y$ the imprint of X in Y by means of morphism φ , and by α the dashed arrow in the corresponding diagram (1.32), i.e.

$$\varphi = \mu \circ \alpha$$

Using the dual reasoning to what we used when proving that β is a strong monomorphism, one can show that α is a strong epimorphism.

5. Consider now a diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \alpha \downarrow & \searrow \mu & \uparrow \beta \\ M & & N \end{array}$$

As we already understood, here α is an epimorphism, hence α is an extension of X with respect to φ . At the same time ε is an envelope of X with respect to φ . Hence there exists a morphism v such that the following diagram is commutative:

$$\begin{array}{ccc} X & & \\ \alpha \downarrow & \searrow \varepsilon & \\ M & \xrightarrow{v} & N \end{array}$$

As a corollary, the following diagram is commutative as well:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \alpha \downarrow & \searrow \varepsilon & \uparrow \beta \\ M & \xrightarrow{v} & N \end{array} \quad (2.18)$$

Similarly, β is a monomorphism, so it is a domain of influence of N in Y by means of φ . At the same time, μ is an imprint of the category \mathbf{K} in the object Y by means of morphism φ . Hence, there exists a morphism v' such that the following diagram is commutative:

$$\begin{array}{ccc} & & Y \\ & \nearrow \mu & \uparrow \beta \\ M & \xrightarrow{v'} & N \end{array}$$

As a corollary, the following diagram is commutative as well:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \alpha \downarrow & \nearrow \mu & \uparrow \beta \\ M & \xrightarrow{v'} & N \end{array} \quad (2.19)$$

From (2.18) and (2.19) we have:

$$\begin{array}{c} \beta \\ \cap \\ \text{Mono} \end{array} \circ v \circ \begin{array}{c} \alpha \\ \cap \\ \text{Epi} \end{array} = \varphi = \begin{array}{c} \beta \\ \cap \\ \text{Mono} \end{array} \circ v' \circ \begin{array}{c} \alpha \\ \cap \\ \text{Epi} \end{array} \implies v = v'$$

I.e. the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \alpha \downarrow & \nearrow \mu & \uparrow \beta \\ M & \xrightarrow{v} & N \end{array}$$

Here $\varepsilon = v \circ \alpha$ is an epimorphism, hence v is an epimorphism as well. On the other hand, $\mu = \beta \circ v$ is a monomorphism, so v is a monomorphism as well. Thus, v is a bimorphism, and $\varphi = \beta \circ v \circ \alpha$ is a nodal decomposition of φ . \square

Envelopes in Epi and imprints of Mono in a category with nodal decomposition.

Theorem 2.8. *Let $\varphi : X \rightarrow Y$ be a morphism in a category \mathbf{K} with nodal decomposition. Then*

- (i) *the epimorphism ε_{\max} in the maximal factorization $\varphi = \mu_{\max} \circ \varepsilon_{\max}$ of φ is an envelope of the object X in \mathbf{K} with respect to the morphism φ :*

$$\text{env}_{\varphi} X = \varepsilon_{\max} = \text{red}_{\infty} \varphi \circ \text{coim}_{\infty} \varphi, \quad \text{Env}_{\varphi} X = \text{Im}_{\infty} \varphi \quad (2.20)$$

- (ii) *the monomorphism μ_{\min} in the minimal factorization $\varphi = \mu_{\min} \circ \varepsilon_{\min}$ of the morphism φ is an imprint of \mathbf{K} on the object Y by means of the morphism φ :*

$$\text{imp}_{\varphi} Y = \mu_{\min} = \text{im}_{\infty} \varphi \circ \text{red}_{\infty} \varphi, \quad \text{Imp}_{\varphi} Y = \text{Coim}_{\infty} \varphi \quad (2.21)$$

Proof. Since (i) and (ii) are dual to each other, it is sufficient to prove (i). The epimorphism $\varepsilon_{\max} = \text{red}_{\infty} \varphi \circ \text{coim}_{\infty} \varphi$ is an extension of X with respect to φ , due to the diagram

$$\begin{array}{ccc} X & \xrightarrow{\text{red}_{\infty} \varphi \circ \text{coim}_{\infty} \varphi} & \text{Im}_{\infty} \varphi \\ & \searrow \varphi & \swarrow \text{im}_{\infty} \varphi \\ & & Y \end{array} \quad (2.22)$$

Let $\sigma : X \rightarrow N$ be another extension of X with respect to φ :

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & N \\ & \searrow \varphi & \swarrow \exists! \nu \\ & Y & \end{array}$$

Then, since σ is an epimorphism, by Theorem 2.1 there exists a morphism $\nu' : N \rightarrow \text{Im}_\infty \varphi$ such that the following diagram is commutative:

$$\begin{array}{ccccc} X & \xrightarrow{\varphi} & Y & & \\ \downarrow \text{coim}_\infty \varphi & \searrow \sigma & \swarrow \nu & \uparrow \text{im}_\infty \varphi & \\ & N & & & \\ \downarrow & \searrow \nu' & & \uparrow & \\ \text{Coim}_\infty \varphi & \xrightarrow{\text{red}_\infty \varphi} & \text{Im}_\infty \varphi & & \end{array}$$

One can transform it into the diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ & \searrow \sigma & \swarrow \nu \\ & N & \\ & \searrow \nu' & \\ & \text{Im}_\infty \varphi & \end{array}$$

$\varepsilon_{\max} = \text{red}_\infty \varphi \circ \text{coim}_\infty \varphi$ (curved arrow from X to $\text{Im}_\infty \varphi$)
 $\text{im}_\infty \varphi$ (curved arrow from Y to $\text{Im}_\infty \varphi$)

which shows that ν' is the required morphism ν from (1.6). \square

Theorem 2.9. *Let \mathbf{K} be a category with nodal decomposition, then*

- (a) *if \mathbf{K} is a category with finite products (respectively, with products over arbitrary index sets), then any object X has an envelope in \mathbf{K} with respect to an arbitrary finite (respectively, to an arbitrary) set of morphisms Φ , among which there is at least one going from X , and*
- (b) *if \mathbf{K} is a category with finite coproducts (respectively, with coproducts over arbitrary index sets), then on any object X there is an imprint of \mathbf{K} by means of an arbitrary finite (respectively, of an arbitrary) set of morphisms Φ , among which there is at least one coming to X .*

Proof. Due to duality, it is sufficient here to prove (a).

1. Consider first the case, when \mathbf{K} is a category with finite products. Let X be an object and Φ a finite set of morphisms. Clearly, it is sufficient to set off a subset $\Phi_X = \{\varphi : X \rightarrow Y_\varphi; \varphi \in \Phi_X\}$ in Φ which consists of the morphisms going from X ,

$$\varphi \in \Phi_X \iff \varphi \in \Phi \ \& \ \text{Dom}(\varphi) = X.$$

Then the envelope with respect to Φ is the same as the envelope with respect to Φ_X . Consider the product of objects $\prod_{\varphi \in \Phi_X} Y_\varphi$ and the corresponding product of morphisms $\prod_{\varphi \in \Phi_X} \varphi : X \rightarrow \prod_{\varphi \in \Phi_X} Y_\varphi$. The envelope of X with respect to the set of morphisms Φ_X is exactly the envelope of X with respect just one morphism $\prod_{\varphi \in \Phi_X} \varphi$. Then we apply Theorem 2.8.

2. Let now \mathbf{K} be a category with products over arbitrary (not necessarily finite) index set. Then the previous reasonings are valid for the case when the set Φ is not necessarily finite. \square

Theorem 2.10. *Let \mathbf{K} be a category with nodal decomposition, then*

- (a) *if \mathbf{K} is a category with products (over arbitrary index sets), then every its object X has an envelope in \mathbf{K} with respect to arbitrary class of morphisms Φ , which contains at least one morphism, going from X , and is generated on the inside by some subset of morphisms, and*
- (b) *if \mathbf{K} is a category with coproducts (over arbitrary index sets), then in any its object X there exists the imprint of \mathbf{K} by means of arbitrary class of morphisms Φ , which contains at least one morphism, coming to X , and is generated on the outside by some subset of morphisms.*

Proof. Again, due to the duality it is sufficient to prove (a). Let $\Psi \subseteq \Phi$ be a subset (not just a class) which generates Φ on the inside. By Theorem 2.9, every object X has envelope with respect to Ψ . On the other hand, by (1.14) this envelope coincides with the envelope with respect to Φ . \square

Theorem 2.11. *Let K be a category with nodal decomposition, then*

- (a) *if K is a co-well-powered category with products (over arbitrary index sets), then any object X has an envelope in K with respect to an arbitrary class of morphisms Φ , among which there is at least one going from X , and*
- (b) *if K is a well-powered category with finite coproducts (over arbitrary index sets), then on any object X there is an imprint of K by means of an arbitrary class of morphisms Φ , among which there is at least one coming to X .*

Proof. Again, due to duality, it is sufficient here to prove (a). Let K be a category with products (over arbitrary index sets), A an object of K , and Φ a class of morphisms in K . The idea of proof is to replace the class Φ by a set of morphism M , such that the envelope with respect to Φ is the same as with respect to M .

Again, like in Theorem 2.9 we can think that Φ consists of morphisms going from X :

$$\forall \varphi \in \Phi \quad \text{Dom}(\varphi) = X.$$

For each $\varphi \in \Phi$ consider a morphism $\varepsilon_\varphi = \text{red}_\infty \varphi \circ \text{coim}_\infty \varphi : X \rightarrow \text{Im}_\infty \varphi$. Let us show that the class Φ can be replaced by the class of morphisms $\{\varepsilon_\varphi; \varphi \in \Phi\}$.

Indeed, suppose $\sigma : X \rightarrow X'$ is an extension of X with respect to morphisms $\{\varepsilon_\varphi; \varphi \in \Phi\}$. Then in the diagram

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X' \\ \varepsilon_\varphi \searrow & & \nearrow \varepsilon' \\ & \text{Im}_\infty \varphi & \\ \downarrow \text{im}_\infty \varphi & & \downarrow \varphi' \\ Y & \xleftarrow{\varphi} & \end{array}$$

the existence of morphism ε' , for which the upper little triangle is commutative, implies the existence of morphism φ' , for which the lower right little triangle is commutative, and since the last (left) little triangle is commutative (being diagram (4.24)), we conclude that the big triangle (the perimeter) is commutative as well. Hence, $\sigma : X \rightarrow X'$ is an extension of X with respect to morphisms Φ .

Conversely, suppose that $\sigma : X \rightarrow X'$ is an extension of X with respect to Φ . Then for any $\varphi \in \Phi$ there exists a morphism φ' such that in the diagram

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X' \\ \varepsilon_\varphi \searrow & & \nearrow \varphi' \\ & \text{Im}_\infty \varphi & \\ \downarrow \text{im}_\infty \varphi & & \downarrow \varphi' \\ Y & \xleftarrow{\varphi} & \end{array}$$

the big triangle (perimeter) is commutative. The lower left little triangle here is commutative as well due to (4.24), hence the following quadrangle is also commutative:

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X' \\ \varepsilon_\varphi \searrow & & \nearrow \varphi' \\ & \text{Im}_\infty \varphi & \\ \downarrow \text{im}_\infty \varphi & & \downarrow \varphi' \\ Y & \xleftarrow{\varphi} & \end{array}$$

Here σ is an epimorphism and $\text{im}_\infty \varphi$ a strong monomorphism (since this is a monomorphism in a strong decomposition of φ). Thus, there exists a diagonal ε' :

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X' \\ & \searrow \varepsilon_\varphi & \swarrow \varepsilon' \\ & \text{Im}_\infty \varphi & \\ & \downarrow \text{im}_\infty \varphi & \\ & Y & \end{array}$$

Here, in particular, the upper triangle is commutative, and, since this is true for any $\varphi \in \Phi$, this means that $\sigma : X \rightarrow X'$ is an extension of X with respect to morphisms $\{\varepsilon_\varphi; \varphi \in \Phi\}$.

Now we see that in the definition of envelope by diagram (1.6) the morphisms ρ and σ are extensions with respect to Φ if and only if they are extension with respect to $\{\varepsilon_\varphi; \varphi \in \Phi\}$. As a corollary, the envelope with respect to Φ is the same as the envelope with respect to $\{\varepsilon_\varphi; \varphi \in \Phi\}$.

Thus, we replaced the class Φ by the class $\{\varepsilon_\varphi; \varphi \in \Phi\}$. Now we recall that all morphisms ε_φ are epimorphisms, and, since our category is co-well-powered, we can find a set M such that every epimorphism ε_φ is isomorphic to some epimorphism $\varepsilon \in M$, i.e. $\varepsilon_\varphi = \iota \circ \varepsilon$ for some isomorphism ι . The set M now replaces the class $\{\varepsilon_\varphi; \varphi \in \Phi\}$ (and thus, a class Φ as well). \square

Theorem 2.12. *Let \mathbf{K} be a category with the nodal decomposition, then*

- (a) *if \mathbf{K} is a co-well-powered category with products (over arbitrary index set), then every object X in \mathbf{K} has an envelope in every class Ω which contains the class **Bim** of all bimorphisms,*

$$\Omega \supseteq \text{Bim},$$

*with respect to arbitrary right ideal of morphisms Φ , which differs morphisms on the outside and has at least one morphism going from X , and the envelope of X in Ω with respect to Φ coincides with the envelopes in **Bim** and in **Epi** with respect to Φ :*

$$\text{env}_\Phi^\Omega X = \text{env}_\Phi^{\text{Bim}} X = \text{env}_\Phi^{\text{Epi}} X.$$

- (b) *if \mathbf{K} is a well-powered category with coproducts (over arbitrary index sets), then in every object X in \mathbf{K} there is an imprint of arbitrary class Ω which contains the class **Bim** of all bimorphisms,*

$$\Omega \supseteq \text{Bim},$$

*by means of an arbitrary left ideal of morphisms Φ which differs morphisms on the inside and contains at least one morphism coming to X , and the imprint in X of the class Ω by means of the class Φ coincides with the envelopes of the classes **Bim** and **Mono** by means of Φ :*

$$\text{imp}_\Phi^\Omega X = \text{imp}_\Phi^{\text{Bim}} X = \text{imp}_\Phi^{\text{Mono}} X.$$

Proof. We prove (a). By Theorem 2.11 there exists the envelope $\text{env}_\Phi^{\text{Epi}} X$. By Theorem 1.2(i) this envelope is a monomorphism, and thus, a bimorphism: $\text{env}_\Phi^{\text{Epi}} X \in \text{Bim}$. Hence, by property 1°(c) on p.20 the envelope in **Epi** must be an envelope in **Bim**: $\text{env}_\Phi^{\text{Epi}} X = \text{env}_\Phi^{\text{Bim}} X$. Now by Theorem 1.3 the envelope in **Bim** must be an envelope in Ω : $\text{env}_\Phi^{\text{Bim}} X = \text{env}_\Phi^\Omega X$. \square

Nets and functorial properties of envelopes in **Epi and of imprints of **Mono**.** The following result (together with Lemma 2.6 below) is a modification of Lemma 1.1 for categories with nodal decomposition.

Lemma 2.2. *Suppose \mathbf{K} is a category with nodal decomposition, $\{X^i, \iota_i^j\}$ a covariant (or contravariant) system there, and $\{\rho^i : X \rightarrow X^i; i \in I\}$ a projective cone from a given object X into $\{X^i, \iota_i^j\}$. If there is a projective limit $\rho = \varprojlim \rho^i : X \rightarrow \varprojlim X^i$, then in its maximal factorization $\rho = \mu_{\max} \circ \varepsilon_{\max}$ the epimorphism ε_{\max} is an envelope of X in the class **Epi** of all epimorphisms with respect to the system of morphisms $\{\rho^i; i \in I\}$:*

$$\text{env}_{\{\rho^i; i \in I\}}^{\text{Epi}} X = \varepsilon_{\max} = \text{red}_\infty \varprojlim \rho^i \circ \text{coim}_\infty \varprojlim \rho^i, \quad \text{Env}_{\{\rho^i; i \in I\}}^{\text{Epi}} X = \text{Im}_\infty \varprojlim \rho^i \quad (2.23)$$

Proof. By definition of projective limit each morphism ρ^j has a continuation π^j at $\varprojlim X^i$. The restriction of π^j at $\text{Im}_\infty \rho$, i.e. the composition $\tau^j = \pi^j \circ \text{im}_\infty \rho$ is a continuation of ρ^j at $\text{Im}_\infty \rho$ along ε_{\max} :

$$\begin{array}{ccccc} X & \xrightarrow{\varepsilon_{\max}} & \text{Im}_\infty \rho & \xrightarrow{\text{im}_\infty \rho} & \varprojlim X^i \\ & \searrow \rho^j & \downarrow \tau^j & \swarrow \pi^j & \\ & & X^j & & \end{array} \quad (2.24)$$

This continuation τ^j is unique since ε_{\max} is an epimorphism, and we can conclude, that the morphism ε_{\max} is an extension of X with respect to the system $\{\rho^i\}$. The other reasoning repeat the step 2 in the proof of Lemma 1.1. \square

An analog of Theorems 1.13 and 1.14 for categories with nodal decomposition is the following:

Theorem 2.13. *Suppose that in a category \mathbf{K} with nodal decomposition there is a net of epimorphisms \mathcal{N} , which generates on the inside a class of morphisms Φ :*

$$\mathcal{N} \subseteq \Phi \subseteq \text{Mor}(\mathbf{K}) \circ \mathcal{N}.$$

Then

- (a) *for each object X in \mathbf{K} a morphism $\text{red}_\infty \varprojlim \mathcal{N}_X \circ \text{coim}_\infty \varprojlim \mathcal{N}_X$ is an envelope $\text{env}_\Phi^{\text{Epi}} X$ in the class Epi of all epimorphisms in \mathbf{K} with respect to the class Φ :*

$$\text{red}_\infty \varprojlim \mathcal{N}_X \circ \text{coim}_\infty \varprojlim \mathcal{N}_X = \text{env}_\Phi^{\text{Epi}} X, \quad (2.25)$$

- (b) *for each morphism $\alpha : X \rightarrow Y$ in \mathbf{K} and for each choice of envelopes $\text{env}_\Phi^{\text{Epi}} X$ and $\text{env}_\Phi^{\text{Epi}} Y$ there is a unique morphism $\text{env}_\Phi^{\text{Epi}} \alpha : \text{Env}_\Phi^{\text{Epi}} X \rightarrow \text{Env}_\Phi^{\text{Epi}} Y$ in \mathbf{K} such that the following diagram is commutative:*

$$\begin{array}{ccc} X & \xrightarrow{\text{env}_\Phi^{\text{Epi}} X} & \text{Env}_\Phi^{\text{Epi}} X \\ \downarrow \alpha & & \downarrow \text{env}_\Phi^{\text{Epi}} \alpha \\ Y & \xrightarrow{\text{env}_\Phi^{\text{Epi}} Y} & \text{Env}_\Phi^{\text{Epi}} Y \end{array} \quad (2.26)$$

- (c) *if in addition the category \mathbf{K} is co-well-powered or strongly well-powered, then the correspondence $(X, \alpha) \mapsto (\text{Env}_\Phi^{\text{Epi}} X, \text{env}_\Phi^{\text{Epi}} \alpha)$ can be defined as a functor from \mathbf{K} into \mathbf{K} :*

$$\text{env}_\Phi^{\text{Epi}}(\beta \circ \alpha) = \text{env}_\Phi^{\text{Epi}} \beta \circ \text{env}_\Phi^{\text{Epi}} \alpha. \quad (2.27)$$

Proof. We follow here the proof of Theorem 1.13, but the difference is that, first, the reference to Lemma 1.1 must be replaced by reference to Lemma 2.2. And, second, since equality $\varprojlim \mathcal{N}_X = \text{Env}_\Phi^{\text{Epi}} Y$ is not valid here, the dotted arrow in diagram (2.26) is constructed in a little bit more complicated way. At the beginning we add diagram (1.56) with the decomposition of the limits $\varprojlim \mathcal{N}_X$ and $\varprojlim \mathcal{N}_Y$ to the components that are important for us:

$$\begin{array}{ccccc} & & \varprojlim \mathcal{N}_X & & \\ & \nearrow & & \searrow & \\ X & \xrightarrow{\text{env}_\Phi^{\text{Epi}} X} & \text{Env}_\Phi^{\text{Epi}} X = \text{Im}_\infty \varprojlim \mathcal{N}_X & \xrightarrow{\text{im}_\infty \varprojlim \mathcal{N}_X} & X_{\mathcal{N}} \\ \downarrow \alpha & & & & \downarrow \alpha_{\mathcal{N}} \\ Y & \xrightarrow{\text{env}_\Phi^{\text{Epi}} Y} & \text{Env}_\Phi^{\text{Epi}} Y = \text{Im}_\infty \varprojlim \mathcal{N}_Y & \xrightarrow{\text{im}_\infty \varprojlim \mathcal{N}_Y} & Y_{\mathcal{N}} \\ & \nwarrow & & \nearrow & \\ & & \varprojlim \mathcal{N}_Y & & \end{array}$$

After that we put $\xi = \alpha_{\mathcal{N}} \circ \text{im}_{\infty} \varprojlim \mathcal{N}_X$ and we obtain a diagram

$$\begin{array}{ccc} X & \xrightarrow{\text{env}_{\Phi}^{\text{Epi}} X} & \text{Env}_{\Phi}^{\text{Epi}} X = \text{Im}_{\infty} \varprojlim \mathcal{N}_X \\ \downarrow \alpha & & \searrow \xi \\ Y & \xrightarrow{\text{env}_{\Phi}^{\text{Epi}} Y} & \text{Env}_{\Phi}^{\text{Epi}} Y = \text{Im}_{\infty} \varprojlim \mathcal{N}_Y \xrightarrow{\text{im}_{\infty} \varprojlim \mathcal{N}_Y} Y_{\mathcal{N}} \end{array}$$

Here the upper horizontal arrow, $\text{env}_{\Phi}^{\text{Epi}} X$, is an epimorphism, and the second lower horizontal arrow, $\text{im}_{\infty} \varprojlim \mathcal{N}_Y$, is a strong monomorphism. So there must exist a morphism ξ' such that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\text{env}_{\Phi}^{\text{Epi}} X} & \text{Env}_{\Phi}^{\text{Epi}} X = \text{Im}_{\infty} \varprojlim \mathcal{N}_X \\ \downarrow \alpha & & \downarrow \xi' \\ Y & \xrightarrow{\text{env}_{\Phi}^{\text{Epi}} Y} & \text{Env}_{\Phi}^{\text{Epi}} Y = \text{Im}_{\infty} \varprojlim \mathcal{N}_Y \xrightarrow{\text{im}_{\infty} \varprojlim \mathcal{N}_Y} Y_{\mathcal{N}} \end{array}$$

and this will be the vertical arrow we need in (2.26).

Finally, third, if we want to define the correspondence $X \mapsto \text{Env}_{\Phi}^{\text{Epi}} X = \text{Im}_{\infty} \varprojlim \mathcal{N}_X$ as a map, we must require that \mathbf{K} is strongly well-powered (and then the passage from $X_{\mathcal{N}}$ to its subobject $\text{Im}_{\infty} \varprojlim \mathcal{N}_X$ can be defined as a map by means of choice axiom), or we must require that \mathbf{K} is co-well-powered (and then the passage from X to its quotient object $\text{red}_{\infty} \varprojlim \mathcal{N}_X \circ \text{coim}_{\infty} \varprojlim \mathcal{N}_X$ can be again defined as a map by means of choice axiom). At the same time, (2.27) will hold since the dotted arrow in (2.26) is unique. \square

The dual results for imprints look as follows:

Lemma 2.3. *Let \mathbf{K} be a category with nodal decomposition, $\{X^i, \iota_i^j\}$ a covariant (or contravariant) system in \mathbf{K} , and $\{\rho^i : X^i \rightarrow X; i \in I\}$ an injective cone from $\{X^i, \iota_i^j\}$ into a given object X . If there is an injective limit $\rho = \varinjlim \rho^i : X \rightarrow \varinjlim X^i$, then in its maximal factorization $\rho = \mu_{\max} \circ \varepsilon_{\max}$ the monomorphism μ_{\max} is an imprint of the class Epi in the object X by means of the class of all monomorphisms with respect to the system of morphisms $\{\rho^i; i \in I\}$:*

$$\text{imp}_{\{\rho^i; i \in I\}}^{\text{Mono}} X = \mu_{\max} = \text{im}_{\infty} \varinjlim \rho^i \circ \text{red}_{\infty} \varinjlim \rho^i, \quad \text{Imp}_{\{\rho^i; i \in I\}}^{\text{Mono}} X = \text{Coim}_{\infty} \varinjlim \rho^i \quad (2.28)$$

Theorem 2.14. *Suppose that in an injectively complete category with nodal decomposition \mathbf{K} there is a net of monomorphisms \mathcal{N} , which generates on the outside a class of morphisms Φ :*

$$\mathcal{N} \subseteq \Phi \subseteq \mathcal{N} \circ \text{Mor}(\mathbf{K}).$$

Then

- (a) *for each object X in \mathbf{K} the morphism $\text{im}_{\infty} \varinjlim \mathcal{N}_X \circ \text{red}_{\infty} \varinjlim \mathcal{N}_X$ is an imprint $\text{imp}_{\Phi}^{\text{Mono}} X$ of the class Mono of all monomorphisms in \mathbf{K} by means of the class Φ :*

$$\text{im}_{\infty} \varinjlim \mathcal{N}_X \circ \text{red}_{\infty} \varinjlim \mathcal{N}_X = \text{imp}_{\Phi}^{\text{Mono}} X, \quad (2.29)$$

- (b) *for each morphism $\alpha : X \rightarrow Y$ in \mathbf{K} and for each choice of imprints $\text{imp}_{\Phi}^{\text{Mono}} X$ and $\text{imp}_{\Phi}^{\text{Mono}} Y$ there is a unique morphism $\text{imp}_{\Phi}^{\text{Mono}} \alpha : \text{imp}_{\Phi}^{\text{Mono}} X \rightarrow \text{imp}_{\Phi}^{\text{Mono}} Y$ in \mathbf{K} such that the following diagram is commutative:*

$$\begin{array}{ccc} X & \xleftarrow{\text{imp}_{\Phi}^{\text{Mono}} X} & \text{imp}_{\Phi}^{\text{Mono}} X \\ \downarrow \alpha & & \downarrow \text{imp}_{\Phi}^{\text{Mono}} \alpha \\ Y & \xleftarrow{\text{imp}_{\Phi}^{\text{Mono}} Y} & \text{imp}_{\Phi}^{\text{Mono}} Y \end{array} \quad (2.30)$$

- (c) *if in addition \mathbf{K} is well-powered or strongly co-well-powered, then the correspondence $(X, \alpha) \mapsto (\text{imp}_{\Phi}^{\text{Mono}} X, \text{imp}_{\Phi}^{\text{Mono}} \alpha)$ can be defined as a functor from \mathbf{K} into \mathbf{K} :*

$$\text{imp}_{\Phi}^{\text{Mono}} 1_X = 1_{\text{imp}_{\Phi}^{\text{Mono}} X}, \quad \text{imp}_{\Phi}^{\text{Mono}} (\beta \circ \alpha) = \text{imp}_{\Phi}^{\text{Mono}} \beta \circ \text{imp}_{\Phi}^{\text{Mono}} \alpha. \quad (2.31)$$

Partial functorial property of envelopes in Epi and imprints of Mono in categories with nodal decomposition. It is instructive to finish the discussion of relations between envelopes, imprints and nodal decomposition by the observation that in the categories with nodal decomposition the envelopes and imprints have some weakened functorial properties. The following two theorems supplement what we already told about this.

- Let us say that a class of morphisms Φ in a category \mathbf{K} is

— *dense on the outside*, if for any object X of this category there is a morphism $\varphi \in \Phi$, going from X :

$$\forall X \in \text{Ob}(\mathbf{K}) \quad \exists \varphi \in \Phi \quad \text{Dom } \varphi = X,$$

— *dense on the inside*, if for any object X of this category there is a morphism $\varphi \in \Phi$, coming to X :

$$\forall X \in \text{Ob}(\mathbf{K}) \quad \exists \varphi \in \Phi \quad \text{Ran } \varphi = X.$$

Let us denote by \mathbf{K}^{Epi} the subcategory in \mathbf{K} with the same objects as in \mathbf{K} , but with epimorphisms of \mathbf{K} as morphisms:

$$\text{Ob}(\mathbf{K}^{\text{Epi}}) = \text{Ob}(\mathbf{K}), \quad \text{Mor}(\mathbf{K}^{\text{Epi}}) = \text{Epi}(\mathbf{K}).$$

Theorem 2.15. *Suppose \mathbf{K} is a co-well-powered category with nodal decomposition and with products (over arbitrary index sets). And let Φ be a dense on the outside class of morphisms in \mathbf{K} satisfying the following condition:*

- (i) *for each epimorphism ε the class $\Phi \circ \varepsilon = \{\varphi \circ \varepsilon; \varphi \in \Phi\}$ is contained in Φ :*

$$\Phi \circ \varepsilon \subseteq \Phi.$$

Then

- (a) *every object X in \mathbf{K} has envelope $\text{Env}_\Phi X$ in \mathbf{K} with respect to the class of morphisms Φ ,*
 (b) *for any epimorphism $\varepsilon : X \rightarrow Y$ in \mathbf{K} there exists a unique morphism $\text{env}_\Phi \varepsilon : \text{Env}_\Phi X \rightarrow \text{Env}_\Phi Y$ such that the following diagram is commutative:*

$$\begin{array}{ccc} X & \xrightarrow{\text{env}_\Phi X} & \text{Env}_\Phi X \\ \downarrow \varepsilon & & \downarrow \text{env}_\Phi \varepsilon \\ Y & \xrightarrow{\text{env}_\Phi Y} & \text{Env}_\Phi Y \end{array} \quad (2.32)$$

- (c) *the correspondence $(X, \varepsilon) \mapsto (\text{Env}_\Phi^{\text{Epi}} X, \text{env}_\Phi^{\text{Epi}} \varepsilon)$ can be defined as a covariant functor from \mathbf{K}^{Epi} into \mathbf{K}^{Epi} :*

$$\text{env}_\Phi^{\text{Epi}} 1_X = 1_{\text{Env}_\Phi^{\text{Epi}} X}, \quad \text{env}_\Phi^{\text{Epi}} (\pi \circ \varepsilon) = \text{env}_\Phi^{\text{Epi}} \pi \circ \text{env}_\Phi^{\text{Epi}} \varepsilon, \quad \varepsilon, \pi \in \text{Epi}(\mathbf{K}).$$

For proof we need the following

Lemma 2.4. *If \mathbf{K} is a co-well-powered category with nodal decomposition and with products (over arbitrary index sets), then for any class of objects \mathbf{L} , for any class of morphisms Φ and for any epimorphism $\varepsilon : X \rightarrow Y$ the following equality holds:*

$$\text{Env}_{\Phi \circ \varepsilon}^{\mathbf{L}} X = \text{Env}_\Phi^{\mathbf{L}} Y \quad (2.33)$$

(i.e. the envelope of X in \mathbf{L} with respect to $\Phi \circ \varepsilon = \{\varphi \circ \varepsilon; \varphi \in \Phi\}$ coincides with the envelope of Y in \mathbf{L} with respect to Φ).

Proof. By property 3° on page 21, there exists a morphism v such that (1.12) is commutative:

$$\begin{array}{ccc} & X & \\ \text{env}_\Phi^{\mathbf{L}} Y \circ \varepsilon \swarrow & & \searrow \text{env}_\Phi^{\mathbf{L}} \varepsilon \circ X \\ \text{Env}_\Phi^{\mathbf{L}} Y & \xrightarrow{v} & \text{Env}_{\Phi \circ \varepsilon}^{\mathbf{L}} X \end{array}$$

Let us show that an inverse morphism also exists. Consider the envelope $\text{env}_\Phi^{\mathbf{L}} Y : Y \rightarrow \text{Env}_\Phi^{\mathbf{L}} Y$, and let us represent it as an envelope with respect to a set of morphisms M , like in the proof of Theorem 2.11. After

that, like in the proof of Theorem 2.9, let us replace the set M by a unique morphism $\psi = \prod_{\chi \in M} \chi$. Then by Theorem 2.8, the envelope with respect to ψ coincides with the nodal image of this morphism:

$$\text{Env}_{\Phi}^L Y = \text{Env}_M^L Y = \text{Env}_{\psi}^L Y = \text{Im}_{\infty} \psi.$$

We obtain a diagram

$$\begin{array}{ccc} X & \xrightarrow{\text{env}_{\Phi \circ \varepsilon}^L X} & \text{Env}_{\Phi \circ \varepsilon}^L X \\ \varepsilon \downarrow & \text{env}_{\Phi}^L Y \circ \varepsilon \searrow & \delta \swarrow \text{dashed} \\ & \text{env}_{\Phi}^L Y = \text{Im}_{\infty} \psi & \\ & \text{env}_{\Phi}^L Y \nearrow & \text{im}_{\infty} \psi \nearrow \\ Y & \xrightarrow{\psi} & B \end{array}$$

where the existence of the morphism δ follows from the fact that $\text{env}_{\Phi \circ \varepsilon}^{\text{Epi}} X$ is an epimorphism, and $\text{im}_{\infty} \psi$ is a strong monomorphism (and thus, the rectangle $(\psi \circ \varepsilon)' \circ \text{env}_{\Phi \circ \varepsilon}^{\text{Epi}} X = \text{im}_{\infty} \psi \circ (\text{env}_{\Phi}^L Y \circ \varepsilon)$ can be divided into two triangles). So we obtain δ such that

$$\begin{array}{ccc} & X & \\ \text{env}_{\Phi}^L Y \circ \varepsilon \swarrow & & \searrow \text{env}_{\Phi \circ \varepsilon}^L X \\ \text{Env}_{\Phi}^L Y & \xleftarrow{\delta} & \text{Env}_{\Phi \circ \varepsilon}^L X \end{array}$$

It remains to verify that v and δ are mutually inverse. First,

$$\delta \circ v \circ \underbrace{\text{env}_{\Phi}^L Y \circ \varepsilon}_{\substack{\cap \\ \text{Epi}}} = \delta \circ \text{env}_{\Phi \circ \varepsilon}^L X = \text{env}_{\Phi}^L Y \circ \varepsilon = 1_{\text{Env}_{\Phi}^L Y} \circ \underbrace{\text{env}_{\Phi}^L Y \circ \varepsilon}_{\substack{\cap \\ \text{Epi}}} \implies \delta \circ v = 1_{\text{Env}_{\Phi}^L Y}.$$

And, second,

$$v \circ \delta \circ \underbrace{\text{env}_{\Phi \circ \varepsilon}^L X}_{\substack{\cap \\ \text{Epi}}} = v \circ \text{env}_{\Phi}^L Y \circ \varepsilon = \text{env}_{\Phi \circ \varepsilon}^L X = 1_{\text{Env}_{\Phi \circ \varepsilon}^L X} \circ \underbrace{\text{env}_{\Phi \circ \varepsilon}^L X}_{\substack{\cap \\ \text{Epi}}} \implies v \circ \delta = 1_{\text{Env}_{\Phi \circ \varepsilon}^L X}.$$

□

Proof of Theorem 2.15. The proposition (a) follows from Theorem 2.11, (c) – from (b) and from the fact that \mathbf{K} is co-well-powered, so we must prove only (b). By Lemma 2.4, $\text{Env}_{\Phi} Y = \text{Env}_{\Phi \circ \varepsilon} X$, and by 1° on page 20, when we pass to the narrower class of morphisms $\Phi \circ \varepsilon \subseteq \Phi$ a dashed arrow appears in the upper triangle of the diagram

$$\begin{array}{ccc} X & \xrightarrow{\text{env}_{\Phi} X} & \text{Env}_{\Phi} X \\ & \searrow \text{env}_{\Phi} Y \circ \varepsilon & \downarrow \text{env}_{\Phi} \varepsilon \\ & & \text{Env}_{\Phi \circ \varepsilon} X \\ \varepsilon \downarrow & & \parallel \\ Y & \xrightarrow{\text{env}_{\Phi} Y} & \text{Env}_{\Phi} Y \end{array}$$

This is the required arrow in (2.32). □

To formulate the dual result let us denote by \mathbf{K}^{Mono} the subcategory in \mathbf{K} with the same objects as in \mathbf{K} , but with monomorphism from \mathbf{K} as morphisms:

$$\text{Ob}(\mathbf{K}^{\text{Mono}}) = \text{Ob}(\mathbf{K}), \quad \text{Mor}(\mathbf{K}^{\text{Mono}}) = \text{Mono}(\mathbf{K}).$$

Theorem 2.16. Suppose \mathbf{K} is a well-powered category with nodal decomposition and with coproducts (over arbitrary index sets). And let Φ be a dense on the inside class of morphisms in \mathbf{K} satisfying the following condition:

(i) for any monomorphism μ the class $\mu \circ \Phi = \{\mu \circ \varphi; \varphi \in \Phi\}$ is contained in Φ :

$$\mu \circ \Phi \subseteq \Phi.$$

Then

- (a) in any object X in \mathbf{K} there is the imprint $\text{Imp}_\Phi X$ of \mathbf{K} by means of the class of objects Φ ,
 (b) for any monomorphism $\mu : X \rightarrow Y$ in \mathbf{K} there is a unique morphism $\text{imp}_\Phi \mu : \text{Imp}_\Phi X \rightarrow \text{Imp}_\Phi Y$ such that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\text{imp}_\Phi X} & \text{Imp}_\Phi X \\ \downarrow \mu & & \downarrow \text{imp}_\Phi \mu \\ Y & \xrightarrow{\text{imp}_\Phi Y} & \text{Imp}_\Phi Y \end{array} \quad (2.34)$$

(c) the correspondence $(X, \mu) \mapsto (\text{Imp}_\Phi^{\text{Mono}} X, \text{imp}_\Phi^{\text{Mono}} \mu)$ can be defined as a covariant functor from \mathbf{K}^{Mono} into \mathbf{K}^{Mono} :

$$\text{imp}_\Phi^{\text{Mono}} 1_X = 1_{\text{Imp}_\Phi^{\text{Mono}} X}, \quad \text{imp}_\Phi^{\text{Mono}} (\nu \circ \mu) = \text{imp}_\Phi^{\text{Mono}} \nu \circ \text{imp}_\Phi^{\text{Mono}} \mu, \quad \mu, \nu \in \text{Mono}(\mathbf{K}).$$

In its proof the following lemma dual to Lemma 2.4 is used:

Lemma 2.5. *If \mathbf{K} is a well-powered category with nodal decomposition and with coproducts (over arbitrary index sets), then for any class of objects \mathbf{L} , for any class of morphisms Φ and for any monomorphism $\mu : X \leftarrow Y$ the following equality holds:*

$$\text{Imp}_{\mu \circ \Phi}^{\mathbf{L}} X = \text{Env}_\Phi^{\mathbf{L}} Y \quad (2.35)$$

(i.e. the imprint of \mathbf{L} in X by means of $\mu \circ \Phi = \{\mu \circ \varphi; \varphi \in \Phi\}$ coincides with the imprint of \mathbf{L} in Y by means of Φ).

(c) Connections with envelopes in SEpi and with imprints of SMono

Nodal decomposition in a category with envelopes in SEpi and imprints of SMono .

Theorem 2.17. *Suppose that in a category \mathbf{K}*

- (a) *strong epimorphisms discern monomorphisms and strong monomorphisms discern epimorphisms (in the sense of definitions on p.50),*
 (b) *each object X has an envelope in the class SEpi of all strong epimorphisms with respect to an arbitrary morphism that goes from X , and, dually, in each object X there is an imprint of the class SMono of all strong monomorphisms by means of an arbitrary morphisms that comes to X .*

Then \mathbf{K} is a category with nodal decomposition.

Proof. Take a morphism $\varphi : X \rightarrow Y$.

1. By condition (b), there is an envelope $\text{env}_\varphi^{\text{SEpi}} X : X \rightarrow \text{Env}_\varphi^{\text{SEpi}} X$ of the object X in the class SEpi of all strong epimorphisms with respect to the morphism φ . Denote by α the morphism that continues φ at $\text{Env}_\varphi^{\text{SEpi}} X$:

$$\begin{array}{ccc} X & & \\ \text{env}_\varphi^{\text{SEpi}} X \downarrow & \searrow \varphi & \\ \text{Env}_\varphi^{\text{SEpi}} X & \xrightarrow{\alpha} & Y \end{array}$$

2. Similarly, by (b) there is an imprint $\text{imp}_\varphi^{\text{SMono}} Y : \text{Imp}_\varphi^{\text{SMono}} Y \rightarrow Y$ of the class SMono of all strong monomorphisms in the object Y by means of φ . Denote by β the morphism that lifts φ to $\text{Imp}_\varphi^{\text{SMono}} X$:

$$\begin{array}{ccc} X & \xrightarrow{\beta} & \text{Imp}_\varphi^{\text{SEpi}} Y \\ & \searrow \varphi & \downarrow \text{imp}_\varphi^{\text{SMono}} Y \\ & & Y \end{array}$$

3. Pasting these triangles together by the common side φ , and throwing away this side, we obtain a quadrangle:

$$\begin{array}{ccc} X & \xrightarrow{\beta} & \text{Imp}_{\varphi}^{\text{SEpi}} Y \\ \text{env}_{\varphi}^{\text{SEpi}} X \downarrow & & \downarrow \text{imp}_{\varphi}^{\text{SMono}} Y \\ \text{Env}_{\varphi}^{\text{SEpi}} X & \xrightarrow{\alpha} & Y \end{array}$$

Here $\text{env}_{\varphi}^{\text{SEpi}} X$ is a strong epimorphism, and $\text{imp}_{\varphi}^{\text{SMono}} Y$ a monomorphism, so there is a diagonal δ

$$\begin{array}{ccc} X & \xrightarrow{\beta} & \text{Imp}_{\varphi}^{\text{SEpi}} Y \\ \text{env}_{\varphi}^{\text{SEpi}} X \downarrow & \nearrow \delta & \downarrow \text{imp}_{\varphi}^{\text{SMono}} Y \\ \text{Env}_{\varphi}^{\text{SEpi}} X & \xrightarrow{\alpha} & Y \end{array} \quad (2.36)$$

Let us show that δ is a bimorphism.

4. Suppose first that δ is not a monomorphism. Then, since the strong epimorphisms discern monomorphisms (by (a)), there is a decomposition $\delta = \delta' \circ \varepsilon$, where ε is a strong epimorphism, which is not an isomorphism. As a corollary, the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \text{env}_{\varphi}^{\text{SEpi}} X \downarrow & \searrow \beta & \uparrow \text{imp}_{\varphi}^{\text{SMono}} Y \\ \text{Env}_{\varphi}^{\text{SEpi}} X & \xrightarrow{\delta} & \text{Imp}_{\varphi}^{\text{SEpi}} Y \\ \varepsilon \downarrow & \nearrow \delta' & \\ M & & \end{array}$$

We see here that the composition $\text{imp}_{\varphi}^{\text{SMono}} Y \circ \delta'$ is a continuation of φ along $\varepsilon \circ \text{env}_{\varphi}^{\text{SEpi}} X$, which in its turn is a strong epimorphism (as a composition of two strong epimorphisms). This means that $\varepsilon \circ \text{env}_{\varphi}^{\text{SEpi}} X$ is an extension of X in the class SEpi with respect to morphism φ . Hence, there is a morphism v from the extension M to the envelope $\text{Env}_{\varphi}^{\text{SEpi}} X$, such that diagram (1.6) is commutative:

$$\begin{array}{ccc} & X & \\ \text{env}_{\varphi}^{\text{SEpi}} X \swarrow & & \searrow \varepsilon \circ \text{env}_{\varphi}^{\text{SEpi}} X \\ \text{Env}_{\varphi}^{\text{SEpi}} X & \xleftarrow{v} & M \end{array}$$

We have now $v \circ \varepsilon \circ \text{env}_{\varphi}^{\text{SEpi}} X = \text{env}_{\varphi}^{\text{SEpi}} X = 1_M \circ \text{env}_{\varphi}^{\text{SEpi}} X$, and, since $\text{env}_{\varphi}^{\text{SEpi}} X$ is an epimorphism, this implies the equality $v \circ \varepsilon = 1_M$, which means that ε is a coretraction. On the other hand, this is an epimorphism, and together this means that ε must be an isomorphism. This contradicts to the choice of ε .

5. Thus, δ must be a monomorphism. By analogy we prove that this is an epimorphism. Let us now add φ to Diagram (2.36) and twist it as follows:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \text{env}_{\varphi}^{\text{SEpi}} X \downarrow & \begin{array}{c} \nearrow \alpha \\ \searrow \beta \end{array} & \uparrow \text{imp}_{\varphi}^{\text{SMono}} Y \\ \text{Env}_{\varphi}^{\text{SEpi}} X & \xrightarrow{\delta} & \text{Imp}_{\varphi}^{\text{SEpi}} Y \end{array}$$

We see now that $\varphi = \text{imp}_{\varphi}^{\text{SMono}} Y \circ \delta \circ \text{env}_{\varphi}^{\text{SEpi}} X$ is a nodal decomposition of φ . □

Envelopes in SEpi and imprints of SMono in a category with nodal decomposition.

Theorem 2.18. *Suppose $\varphi : X \rightarrow Y$ is a morphism in a category \mathbf{K} with nodal decomposition. Then*

- (i) its nodal coimage $\text{coim}_\infty \varphi$ is an envelope of X in the class SEpi of all strong epimorphisms with respect to the morphism φ :

$$\text{env}_\varphi^{\text{SEpi}} X = \text{coim}_\infty \varphi, \quad \text{Env}_\varphi^{\text{SEpi}} X = \text{Coim}_\infty \varphi \quad (2.37)$$

- (ii) its nodal image $\text{im}_\infty \varphi$ is an imprint of the class SMono of all strong monomorphisms in Y by means of the morphism φ :

$$\text{imp}_\varphi^{\text{Mono}} Y = \text{im}_\infty \varphi, \quad \text{Imp}_\varphi^{\text{Mono}} Y = \text{Im}_\infty \varphi \quad (2.38)$$

Proof. Due to duality between (i) and (ii) it is sufficient to prove (i). Since $\text{coim}_\infty \varphi \in \text{SEpi}$, the morphism $\text{coim}_\infty \varphi : X \rightarrow \text{Coim}_\infty \varphi$ can be treated as an extension of X in the class SEpi with respect to φ :

$$\begin{array}{ccc} X & \xrightarrow{\text{coim}_\infty \varphi} & \text{Coim}_\infty \varphi \\ & \searrow \varphi & \swarrow \text{im}_\infty \varphi \circ \text{red}_\infty \varphi \\ & Y & \end{array}$$

Suppose $\sigma : X \rightarrow X'$ is another extension:

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X' \\ & \searrow \varphi & \swarrow \nu \\ & Y & \end{array}$$

Let us put it into the diagram of nodal decomposition for φ :

$$\begin{array}{ccccc} X & \xrightarrow{\varphi} & & & Y \\ & \searrow \sigma & & \swarrow \nu & \\ & X' & & & \\ \text{coim}_\infty \varphi \downarrow & & & & \uparrow \text{im}_\infty \varphi \\ \text{Coim}_\infty \varphi & \xrightarrow{\text{red}_\infty \varphi} & \text{Im}_\infty \varphi & & \end{array}$$

Since σ is an epimorphism, by (2.6) there is a morphism ν' such that the following diagram is commutative:

$$\begin{array}{ccccc} X & \xrightarrow{\varphi} & & & Y \\ & \searrow \sigma & & \swarrow \nu & \\ & X' & & & \\ \text{coim}_\infty \varphi \downarrow & & & & \uparrow \text{im}_\infty \varphi \\ \text{Coim}_\infty \varphi & \xrightarrow{\text{red}_\infty \varphi} & \text{Im}_\infty \varphi & & \end{array}$$

Further, in the quadrangle $\nu' \circ \sigma = \text{red}_\infty \varphi \circ \text{coim}_\infty \varphi$ the morphism σ is a strong epimorphism, and $\text{red}_\infty \varphi$ a monomorphism, so there exists a (unique) diagonal ν :

$$\begin{array}{ccccc} X & \xrightarrow{\varphi} & & & Y \\ & \searrow \sigma & & \swarrow \nu & \\ & X' & & & \\ \text{coim}_\infty \varphi \downarrow & & & & \uparrow \text{im}_\infty \varphi \\ \text{Coim}_\infty \varphi & \xrightarrow{\text{red}_\infty \varphi} & \text{Im}_\infty \varphi & & \end{array}$$

Here the left little triangle is a variant of diagram (1.6) with $\rho = \text{coim}_\infty \varphi$. □

Of the following three theorems the first two are proved exactly like Theorems 2.9 and 2.10, and the proof of the third one is a little modification of that one of Theorem 2.11:

Theorem 2.19. *Suppose \mathbf{K} is a category with nodal decomposition, then*

- (a) *if \mathbf{K} is a category with finite products (respectively, with products over arbitrary index sets), then each object X in \mathbf{K} has an envelope in the class \mathbf{SEpi} of all strong epimorphisms with respect to arbitrary finite (respectively, to arbitrary, not necessarily finite) set of morphisms Φ , among which there is at least one going from X , and*
- (b) *if \mathbf{K} is a category with finite co-products (respectively, with co-products over arbitrary index sets), then in each object X in \mathbf{K} there is an imprint of the class \mathbf{SMono} of all strong monomorphisms by means of arbitrary finite (respectively, of arbitrary, not necessarily finite) set of morphisms Φ , among which there is at least one coming to X .*

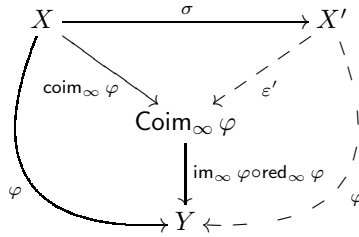
Theorem 2.20. *Suppose \mathbf{K} is a category with nodal decomposition, then*

- (a) *if \mathbf{K} is a category with products (over arbitrary index sets), then each object X in \mathbf{K} has an envelope in the class \mathbf{SEpi} of all strong epimorphisms with respect to arbitrary class of morphisms Φ , which has at least one going to X , and contains a set of morphisms that generates Φ on the inside, and*
- (b) *if \mathbf{K} is a category with co-products (over arbitrary index sets), then in each object X in \mathbf{K} there is an imprint of the class \mathbf{SMono} of all strong monomorphisms by means of arbitrary class of morphisms Φ , which has at least one coming to X , and contains a set of morphisms that generates Φ on the outside.*

Theorem 2.21. *Suppose \mathbf{K} is a category with nodal decomposition, then*

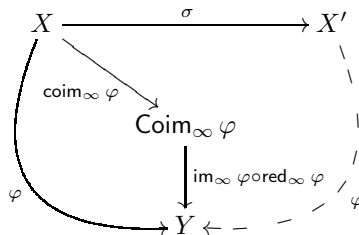
- (a) *if \mathbf{K} is a co-well-powered category with products (over arbitrary index sets), then each object X in \mathbf{K} has an envelope in the class \mathbf{SEpi} of all strong epimorphisms with respect to arbitrary class of morphisms Φ , among which there is at least one going from X , and*
- (b) *if \mathbf{K} is a well-powered category with co-products (over arbitrary index sets), then in each object X in \mathbf{K} there is an imprint of the class \mathbf{SMono} of all strong monomorphisms by means of arbitrary class of morphisms Φ , among which there is at least one coming to X .*

Proof. The difference with the proof of Theorem 2.11 is that the morphisms $\varphi \in \Phi$ are replaced by morphisms $\text{coim}_\infty \varphi : X \rightarrow \text{Coim}_\infty \varphi$. If $\sigma : X \rightarrow X'$ is an extension of X with respect to morphisms $\{\text{coim}_\infty \varphi; \varphi \in \Phi\}$, then in the diagram



the existence of the morphism ε' such that the upper left triangle is commutative, implies the existence of the morphism φ' such that the right lower triangle is commutative, and, since the rest little triangle (the lower left) is commutative, being diagram (4.24), we conclude that the big triangle (perimeter) is also commutative. Thus, $\sigma : X \rightarrow X'$ is an extension of X with respect to Φ .

On the contrary, if $\sigma : X \rightarrow X'$ is an extension of X with respect to morphisms Φ , then for each $\varphi \in \Phi$ there is a morphism φ' such that in the diagram



the big triangle (perimeter) is commutative. The lower left little triangle here is also commutative, by (4.24),

hence the quadrangle

$$\begin{array}{ccc}
 X & \xrightarrow{\sigma} & X' \\
 \searrow \text{coim}_\infty \varphi & & \vdots \\
 & \text{Coim}_\infty \varphi & \\
 \downarrow \text{im}_\infty \varphi \circ \text{red}_\infty \varphi & & \vdots \\
 Y & \xleftarrow{\varphi'} &
 \end{array}$$

is also commutative. Here σ is a strong epimorphism, and $\text{im}_\infty \varphi \circ \text{red}_\infty \varphi$ a monomorphism (as a composition of two monomorphisms). Hence, there exists (and is unique, since σ is an epimorphism) a diagonal ε' :

$$\begin{array}{ccc}
 X & \xrightarrow{\sigma} & X' \\
 \searrow \text{coim}_\infty \varphi & \swarrow \varepsilon' & \vdots \\
 & \text{Coim}_\infty \varphi & \\
 \downarrow \text{im}_\infty \varphi \circ \text{red}_\infty \varphi & & \vdots \\
 Y & \xleftarrow{\varphi'} &
 \end{array}$$

In particular, the upper triangle is commutative, and, since this is true for each $\varphi \in \Phi$, this means that $\sigma : X \rightarrow X'$ is an extension of X with respect to morphism $\{\text{coim}_\infty \varphi; \varphi \in \Phi\}$.

We obtain that in the definition of the envelope by diagram (1.6) the morphisms ρ and σ belong to the class of extensions with respect to Φ if and only if they belong to class of extensions with respect to $\{\text{coim}_\infty \varphi; \varphi \in \Phi\}$. We conclude that the envelope with respect to Φ is the same as the envelope with respect to $\{\text{coim}_\infty \varphi; \varphi \in \Phi\}$.

Thus, we replaced the class Φ with the class $\{\text{coim}_\infty \varphi; \varphi \in \Phi\}$. Now we must recall that all the morphisms $\text{coim}_\infty \varphi$ are strong epimorphisms, and since our category is co-well-powered, we can find a set M among them, such that any epimorphism $\text{coim}_\infty \varphi$ is isomorphic to some epimorphism $\varepsilon \in M$. The set M now replaces the class $\{\text{coim}_\infty \varphi; \varphi \in \Phi\}$ (and thus, the class Φ), and further we apply Theorem 2.19. \square

Nets and the functorial properties of envelopes in SEpi and of imprints of SMono. The following result (together with Lemma 2.2 above) is a modification of Lemma 1.1 for categories with nodal decomposition.

Lemma 2.6. *Suppose K is a category with nodal decomposition, $\{X^i, \iota_i^j\}$ a covariant (or contravariant) system there, and $\{\rho^i : X \rightarrow X^i; i \in I\}$ a projective cone from a given object X into $\{X^i, \iota_i^j\}$. If there is a projective limit $\rho = \varprojlim \rho^i : X \rightarrow \varprojlim X^i$, then its strong coimage $\text{coim}_\infty \rho$ is an envelope of X in the class SEpi of all strong epimorphisms with respect to the system of morphisms $\{\rho^i; i \in I\}$:*

$$\text{env}_{\{\rho^i; i \in I\}}^{\text{SEpi}} X = \text{coim}_\infty \varprojlim \rho^i, \quad \text{Env}_{\{\rho^i; i \in I\}}^{\text{SEpi}} X = \text{Coim}_\infty \varprojlim \rho^i \quad (2.39)$$

Proof. By definition of projective limit, each morphism ρ^j has a continuation π^j at $\varprojlim X^i$. The restriction of π^j at $\text{Im}_\infty \rho$, i.e. the composition $\tau^j = \pi^j \circ \text{im}_\infty \rho \circ \text{red}_\infty \rho$ is a continuation of ρ^j at $\text{Coim}_\infty \rho$ along $\text{coim}_\infty \rho$:

$$\begin{array}{ccccc}
 X & \xrightarrow{\varepsilon_{\max}} & \text{Coim}_\infty \rho & \xrightarrow{\text{im}_\infty \rho \circ \text{red}_\infty \rho} & \varprojlim X^i \\
 \searrow \rho^j & & \downarrow \tau^j & & \vdots \\
 & & X^j & \xleftarrow{\pi^j} &
 \end{array} \quad (2.40)$$

This continuation τ^j is unique since $\text{coim}_\infty \rho$ is an epimorphism, and we can conclude that the morphism $\text{coim}_\infty \rho$ is an extension of X with respect to the system $\{\rho^i\}$. The rest reasoning repeat step 2 of the proof of Lemma 1.1. \square

The following fact is proved by analogy with Theorem 2.13:

Theorem 2.22. *Suppose in a category K with nodal decomposition there is a net of epimorphisms \mathcal{N} , which generates on the inside a class of morphisms Φ :*

$$\mathcal{N} \subseteq \Phi \subseteq \text{Mor}(K) \circ \mathcal{N}.$$

Then

- (a) for each object X in \mathbf{K} the morphism $\text{coim}_\infty \varprojlim \mathcal{N}_X$ is an envelope $\text{env}_\Phi^{\text{SEpi}} X$ in the class SEpi of strong epimorphisms in \mathbf{K} with respect to the class Φ :

$$\text{coim}_\infty \varprojlim \mathcal{N}_X = \text{env}_\Phi^{\text{SEpi}} X, \quad (2.41)$$

- (b) for each morphism $\alpha : X \rightarrow Y$ in \mathbf{K} and for each choice of envelopes $\text{Env}_\Phi^{\text{SEpi}} X$ and $\text{Env}_\Phi^{\text{SEpi}} Y$ there is a unique morphism $\text{env}_\Phi^{\text{SEpi}} \alpha : \text{Env}_\Phi^{\text{SEpi}} X \rightarrow \text{Env}_\Phi^{\text{SEpi}} Y$ in \mathbf{K} such that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\text{env}_\Phi^{\text{SEpi}} X} & \text{Env}_\Phi^{\text{SEpi}} X \\ \downarrow \alpha & & \downarrow \text{env}_\Phi^{\text{SEpi}} \alpha \\ Y & \xrightarrow{\text{env}_\Phi^{\text{SEpi}} Y} & \text{Env}_\Phi^{\text{SEpi}} Y \end{array} \quad (2.42)$$

- (c) if in addition the category \mathbf{K} is well-powered or strongly co-well-powered, then the correspondence $(X, \alpha) \mapsto (\text{Env}_\Phi^{\text{SEpi}} X, \text{env}_\Phi^{\text{SEpi}} \alpha)$ can be defined as a covariant functor from \mathbf{K} into \mathbf{K} :

$$\text{env}_\Phi^{\text{SEpi}} 1_X = 1_{\text{Env}_\Phi^{\text{SEpi}} X}, \quad \text{env}_\Phi^{\text{SEpi}} (\beta \circ \alpha) = \text{env}_\Phi^{\text{SEpi}} \beta \circ \text{env}_\Phi^{\text{SEpi}} \alpha. \quad (2.43)$$

Proof. Here the difference with the proof of Theorem 2.13 is in details of constructing the dotted arrow in (2.42). We need, first, to add diagram (1.56) with the decompositions of limits $\varprojlim \mathcal{N}_X$ and $\varprojlim \mathcal{N}_Y$ as follows:

$$\begin{array}{ccccc} X & \xrightarrow{\text{env}_\Phi^{\text{SEpi}} X = \text{coim}_\infty \varprojlim \mathcal{N}_X} & \text{Env}_\Phi^{\text{SEpi}} X = \text{Coim}_\infty \varprojlim \mathcal{N}_X & \xrightarrow{\text{im}_\infty \varprojlim \mathcal{N}_X \circ \text{red}_\infty \varprojlim \mathcal{N}_X} & X_{\mathcal{N}} \\ \downarrow \alpha & & \downarrow \text{env}_\Phi^{\text{SEpi}} \alpha & & \downarrow \alpha_{\mathcal{N}} \\ Y & \xrightarrow{\text{env}_\Phi^{\text{SEpi}} Y = \text{coim}_\infty \varprojlim \mathcal{N}_Y} & \text{Env}_\Phi^{\text{SEpi}} Y = \text{Coim}_\infty \varprojlim \mathcal{N}_Y & \xrightarrow{\text{im}_\infty \varprojlim \mathcal{N}_Y \circ \text{red}_\infty \varprojlim \mathcal{N}_Y} & Y_{\mathcal{N}} \end{array}$$

$\varprojlim \mathcal{N}_X$ (top curved arrow), $\varprojlim \mathcal{N}_Y$ (bottom curved arrow)

After that we put $\xi = \alpha_{\mathcal{N}} \circ \text{im}_\infty \varprojlim \mathcal{N}_X \circ \text{red}_\infty \varprojlim \mathcal{N}_X$, and we obtain the diagram

$$\begin{array}{ccc} X & \xrightarrow{\text{env}_\Phi^{\text{SEpi}} X = \text{coim}_\infty \varprojlim \mathcal{N}_X} & \text{Env}_\Phi^{\text{SEpi}} X = \text{Im}_\infty \varprojlim \mathcal{N}_X \\ \downarrow \alpha & & \downarrow \text{env}_\Phi^{\text{SEpi}} \alpha \\ Y & \xrightarrow{\text{env}_\Phi^{\text{SEpi}} Y = \text{coim}_\infty \varprojlim \mathcal{N}_Y} & \text{Env}_\Phi^{\text{SEpi}} Y = \text{Im}_\infty \varprojlim \mathcal{N}_Y \end{array}$$

$\xrightarrow{\xi} Y_{\mathcal{N}}$ (from $\text{Env}_\Phi^{\text{SEpi}} X$)
 $\xrightarrow{\text{im}_\infty \varprojlim \mathcal{N}_Y \circ \text{red}_\infty \varprojlim \mathcal{N}_Y} Y_{\mathcal{N}}$ (from $\text{Env}_\Phi^{\text{SEpi}} Y$)

Here the upper horizontal arrow, $\text{env}_\Phi^{\text{SEpi}} X = \text{coim}_\infty \varprojlim \mathcal{N}_X$, is a strong epimorphism, and the second lower horizontal arrow, $\text{im}_\infty \varprojlim \mathcal{N}_Y \circ \text{red}_\infty \varprojlim \mathcal{N}_Y$, is a monomorphism. Hence there must exist a morphism ξ' such that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\text{env}_\Phi^{\text{SEpi}} X = \text{coim}_\infty \varprojlim \mathcal{N}_X} & \text{Env}_\Phi^{\text{SEpi}} X = \text{Im}_\infty \varprojlim \mathcal{N}_X \\ \downarrow \alpha & & \downarrow \xi' \\ Y & \xrightarrow{\text{env}_\Phi^{\text{SEpi}} Y = \text{coim}_\infty \varprojlim \mathcal{N}_Y} & \text{Env}_\Phi^{\text{SEpi}} Y = \text{Im}_\infty \varprojlim \mathcal{N}_Y \end{array}$$

$\xrightarrow{\xi} Y_{\mathcal{N}}$ (from $\text{Env}_\Phi^{\text{SEpi}} X$)
 $\xrightarrow{\text{im}_\infty \varprojlim \mathcal{N}_Y \circ \text{red}_\infty \varprojlim \mathcal{N}_Y} Y_{\mathcal{N}}$ (from $\text{Env}_\Phi^{\text{SEpi}} Y$)

and this will be the missing vertical arrow in (2.42).

Besides this, when constructing the functor $\alpha \mapsto \text{env}_\Phi^{\text{SEpi}} \alpha$, in contrast to Theorem 2.13, we must require that \mathbf{K} is well-powered (and then the passage from the object $X_{\mathcal{N}}$ to the subobject $\text{im}_\infty \varprojlim \mathcal{N}_Y \circ \text{red}_\infty \varprojlim \mathcal{N}_Y$ can be organized as a map with the help of the choice axiom), or that \mathbf{K} is strongly co-well-powered (and then the passage from X to the quotient object $\text{Coim}_\infty \varprojlim \mathcal{N}_X$ can be organized as a map, again with the help of the choice axiom). And (2.43) will be true because of the uniqueness of the dotted arrow in (2.42). \square

The dual propositions are the following.

Lemma 2.7. *Let \mathbf{K} be a category with nodal decomposition, $\{X^i, \iota_i^j\}$ a covariant (or contravariant) system in \mathbf{K} , and $\{\rho^i : X \leftarrow X^i; i \in I\}$ and injective cone from $\{X^i, \iota_i^j\}$ into a given object X . If there is an injective limit $\rho = \varinjlim \rho^i : X \leftarrow \varinjlim X^i$, then its nodal image $\text{im}_\infty \rho$ is an imprint in X of the class SMono of all strong monomorphisms by means of the system of morphisms $\{\rho^i; i \in I\}$:*

$$\text{imp}_{\{\rho^i; i \in I\}}^{\text{SMono}} X = \text{im}_\infty \varinjlim \rho^i, \quad \text{Imp}_{\{\rho^i; i \in I\}}^{\text{SMono}} X = \text{Im}_\infty \varinjlim \rho^i \quad (2.44)$$

Theorem 2.23. *Suppose in a category \mathbf{K} with nodal decomposition there is a net \mathcal{N} which generates on the outside a class of morphisms Φ :*

$$\mathcal{N} \subseteq \Phi \subseteq \mathcal{N} \circ \text{Mor}(\mathbf{K}).$$

Then

- (a) *for each object X in \mathbf{K} the morphism $\text{im}_\infty \varinjlim \mathcal{N}_X$ is an imprint $\text{imp}_\Phi^{\text{SMono}} X$ of the class SMono of strong monomorphisms of the category \mathbf{K} by means of the class of morphisms Φ :*

$$\text{im}_\infty \varinjlim \mathcal{N}_X = \text{imp}_\Phi^{\text{SMono}} X, \quad (2.45)$$

- (b) *for each morphism $\alpha : X \rightarrow Y$ in \mathbf{K} and for each choice of imprints $\text{imp}_\Phi^{\text{SMono}} X$ and $\text{imp}_\Phi^{\text{SMono}} Y$ there is a unique morphism $\text{imp}_\Phi^{\text{SMono}} \alpha : \text{imp}_\Phi^{\text{SMono}} X \rightarrow \text{imp}_\Phi^{\text{SMono}} Y$ in \mathbf{K} such that the following diagram is commutative:*

$$\begin{array}{ccc} X & \xleftarrow{\text{imp}_\Phi^{\text{SMono}} X} & \text{imp}_\Phi^{\text{SMono}} X \\ \downarrow \alpha & & \downarrow \text{imp}_\Phi^{\text{SMono}} \alpha \\ Y & \xleftarrow{\text{imp}_\Phi^{\text{SMono}} Y} & \text{imp}_\Phi^{\text{SMono}} Y \end{array} \quad (2.46)$$

- (c) *if in addition \mathbf{K} is co-well-powered or strongly well-powered, then the correspondence $(X, \alpha) \mapsto (\text{Imp}_\Phi^{\text{SMono}} X, \text{imp}_\Phi^{\text{SMono}} \alpha)$ can be defined as a covariant functor from \mathbf{K} into \mathbf{K} :*

$$\text{imp}_\Phi^{\text{SMono}} 1_X = 1_{\text{Imp}_\Phi^{\text{SMono}} X}, \quad \text{imp}_\Phi^{\text{SMono}} (\beta \circ \alpha) = \text{imp}_\Phi^{\text{SMono}} \beta \circ \text{imp}_\Phi^{\text{SMono}} \alpha. \quad (2.47)$$

Partial functorial property of envelopes in SEpi and imprints in SMono in the categories with nodal decomposition. The following results are analogous to Theorems 2.15 and 2.16.

Theorem 2.24. *Suppose \mathbf{K} is a strongly co-well-powered category with nodal decomposition, and with products (over arbitrary index set). Let Φ be a dense on the outside class of morphisms in \mathbf{K} such that*

- (i) *for each epimorphism ε the class $\Phi \circ \varepsilon = \{\varphi \circ \varepsilon; \varphi \in \Phi\}$ is contained in Φ :*

$$\Phi \circ \varepsilon \subseteq \Phi.$$

Then

- (a) *each object X in \mathbf{K} has an envelope $\text{Env}_\Phi^{\text{SEpi}} X$ in the class SEpi of all strong epimorphisms with respect to the class Φ ,*
- (b) *for each epimorphism $\varepsilon : X \rightarrow Y$ in \mathbf{K} there is a unique morphism $\text{env}_\Phi^{\text{SEpi}} \varepsilon : \text{Env}_\Phi^{\text{SEpi}} X \rightarrow \text{Env}_\Phi^{\text{SEpi}} Y$ such that the following diagram is commutative:*

$$\begin{array}{ccc} X & \xrightarrow{\text{env}_\Phi^{\text{SEpi}} X} & \text{Env}_\Phi^{\text{SEpi}} X \\ \downarrow \varepsilon & & \downarrow \text{env}_\Phi^{\text{SEpi}} \varepsilon \\ Y & \xrightarrow{\text{env}_\Phi^{\text{SEpi}} Y} & \text{Env}_\Phi^{\text{SEpi}} Y \end{array} \quad (2.48)$$

- (c) *the correspondence $(X, \varepsilon) \mapsto (\text{Env}_\Phi^{\text{SEpi}} X, \text{env}_\Phi^{\text{SEpi}} \varepsilon)$ can be defined as a covariant functor from \mathbf{K}^{Epi} into \mathbf{K}^{Epi} :*

$$\text{env}_\Phi^{\text{SEpi}} 1_X = 1_{\text{Env}_\Phi^{\text{SEpi}} X}, \quad \text{env}_\Phi^{\text{SEpi}} (\pi \circ \varepsilon) = \text{env}_\Phi^{\text{SEpi}} \pi \circ \text{env}_\Phi^{\text{SEpi}} \varepsilon, \quad \varepsilon, \pi \in \text{Epi}(\mathbf{K}).$$

This is proved by the same reasoning as in the proof of Theorem 2.15, but Lemma 2.4 must be replaced by the following one:

Lemma 2.8. *If \mathbf{K} is a strongly co-well-powered category with products (over arbitrary index sets), then for each class of morphisms Φ and for each epimorphism $\varepsilon : X \rightarrow Y$ in \mathbf{K}*

$$\text{Env}_{\Phi \circ \varepsilon}^{\text{SEpi}} X = \text{Env}_{\Phi}^{\text{SEpi}} Y \quad (2.49)$$

(i.e. the envelope of X in SEpi with respect to the class $\Phi \circ \varepsilon = \{\varphi \circ \varepsilon; \varphi \in \Phi\}$ coincides with the envelope of Y in SEpi with respect to Φ).

Proof. Here one must follow the proof of Lemma 2.4 with the exception of the moment where the existence of morphism δ is proved. The corrections are the following. We consider the envelope $\text{env}_{\Phi}^{\text{SEpi}} Y : Y \rightarrow \text{Env}_{\Phi}^{\text{SEpi}} Y$. Like in Lemma 2.4, we have to represent it as an envelope with respect to some set of morphisms M (the trick used in the proof of Theorem 2.11), and then (like in the proof of Theorem 2.9) we replace the set M by the unique morphism $\psi = \prod_{\chi \in M} \chi$. Then by Theorem 2.18 the envelope with respect to ψ will be the nodal coimage of this morphism:

$$\text{Env}_{\Phi}^{\text{SEpi}} Y = \text{Env}_M^{\text{SEpi}} Y = \text{Env}_{\psi}^{\text{SEpi}} Y = \text{Coim}_{\infty} \psi.$$

After that a diagram appears

$$\begin{array}{ccc}
 X & \xrightarrow{\text{env}_{\Phi \circ \varepsilon}^{\text{SEpi}} X} & \text{Env}_{\Phi \circ \varepsilon}^{\text{SEpi}} X \\
 \downarrow \varepsilon & \searrow \text{env}_{\Phi}^{\text{SEpi}} Y \circ \varepsilon & \downarrow (\psi \circ \varepsilon)' \\
 & \text{Env}_{\Phi}^{\text{SEpi}} Y = \text{Coim}_{\infty} \psi & \\
 & \swarrow \text{env}_{\Phi}^{\text{SEpi}} Y & \swarrow \text{im}_{\infty} \psi \circ \text{red}_{\infty} \psi \\
 Y & \xrightarrow{\psi} & B
 \end{array}$$

(Note: A dashed arrow δ points from $\text{Env}_{\Phi}^{\text{SEpi}} Y$ to $\text{Env}_{\Phi \circ \varepsilon}^{\text{SEpi}} X$.)

Here the existence of the morphism δ follows from the fact that $\text{env}_{\Phi \circ \varepsilon}^{\text{SEpi}} X$ is a strong epimorphism, and $\text{im}_{\infty} \psi \circ \text{red}_{\infty} \psi$ a monomorphism (and thus the quadrangle $(\psi \circ \varepsilon)' \circ \text{env}_{\Phi \circ \varepsilon}^{\text{SEpi}} X = (\text{im}_{\infty} \psi \circ \text{red}_{\infty} \psi) \circ (\text{env}_{\Phi}^{\text{SEpi}} Y \circ \varepsilon)$ can be splitted into triangles). After that we repeat the reasoning of proof of Lemma 2.4. \square

The dual result is the following:

Theorem 2.25. *Suppose \mathbf{K} is a strongly well-powered category with nodal decomposition, and co-products (over arbitrary index sets). Let Φ be a dense on the inside class of morphisms in \mathbf{K} such that*

(i) *for each monomorphism μ the class $\mu \circ \Phi = \{\mu \circ \varphi; \varphi \in \Phi\}$ is contained in Φ :*

$$\mu \circ \Phi \subseteq \Phi.$$

Then

- (a) *in each object X of \mathbf{K} there is an imprint $\text{Imp}_{\Phi}^{\text{SMono}} X$ of the class SMono of all monomorphisms by means of the class of morphisms Φ ,*
- (b) *for each monomorphism $\mu : X \rightarrow Y$ in \mathbf{K} there is a unique morphism $\text{imp}_{\Phi}^{\text{SMono}} \mu : \text{Imp}_{\Phi}^{\text{SMono}} X \rightarrow \text{Imp}_{\Phi}^{\text{SMono}} Y$ such that the following diagram is commutative:*

$$\begin{array}{ccc}
 X & \xrightarrow{\text{imp}_{\Phi}^{\text{SMono}} X} & \text{Imp}_{\Phi}^{\text{SMono}} X \\
 \downarrow \mu & & \downarrow \text{imp}_{\Phi}^{\text{SMono}} \mu \\
 Y & \xrightarrow{\text{imp}_{\Phi}^{\text{SMono}} Y} & \text{Imp}_{\Phi}^{\text{SMono}} Y
 \end{array} \quad (2.50)$$

- (c) *the correspondence $(X, \mu) \mapsto (\text{Imp}_{\Phi}^{\text{SMono}} X, \text{imp}_{\Phi}^{\text{SMono}} \mu)$ can be defined as a covariant functor from \mathbf{K}^{Mono} into \mathbf{K}^{Mono} :*

$$\text{imp}_{\Phi}^{\text{SMono}} 1_X = 1_{\text{Imp}_{\Phi}^{\text{SMono}} X}, \quad \text{imp}_{\Phi}^{\text{SMono}} (\nu \circ \mu) = \text{imp}_{\Phi}^{\text{SMono}} \nu \circ \text{imp}_{\Phi}^{\text{SMono}} \mu, \quad \mu, \nu \in \text{Mono}(\mathbf{K}).$$

And the corresponding dual lemma looks as follows:

Lemma 2.9. *If \mathbf{K} is a strongly well-powered category with nodal decomposition and co-products (over arbitrary index sets), then for each class of morphisms Φ and for each monomorphism $\mu : X \leftarrow Y$ in \mathbf{K} the following formula holds:*

$$\text{Imp}_{\mu \circ \Phi}^{\text{SMono}} X = \text{Env}_{\Phi}^{\text{SMono}} Y \quad (2.51)$$

(the imprint of SMono in X by means of the class of morphisms $\mu \circ \Phi = \{\mu \circ \varphi; \varphi \in \Phi\}$ coincides with the imprint of SMono in Y with respect to the class Φ).

§ 3 The category of stereotype spaces \mathbf{Ste}

Stereotype spaces were considered in [2] (see also [3]) and are defined as follows. Let X be a locally convex space over \mathbb{C} . Denote by X^* the space of all linear continuous functionals $f : X \rightarrow \mathbb{C}$ endowed with the topology of uniform convergence on totally bounded sets in X . Then X is said to be *stereotype*, if the natural map

$$i_X : X \rightarrow (X^*)^* \quad | \quad i_X(x)(f) = f(x), \quad x \in X, f \in X^*$$

is an isomorphism of locally convex spaces. The class of stereotype spaces \mathbf{Ste} is very wide, since it contains all quasicomplete barreled spaces (in particular, all Banach and all Fréchet spaces). Besides this, \mathbf{Ste} forms a category (with linear continuous maps as morphisms). It was noticed in [2] that this category is complete. Moreover, since the passage to the injective and projective limits in \mathbf{Ste} (as well as in the category \mathbf{LCS} of locally convex spaces) is described as a standard operation (a map which assigns to each covariant or contravariant system its injective and projective limits), we come to the following proposition:

Theorem 3.1. *The category \mathbf{Ste} of stereotype spaces is determined.*

(a) Variations of openness and closure of morphisms

Open and closed morphisms.

- Let us say that a linear continuous map of locally convex spaces $\varphi : X \rightarrow Y$ is *open*, if the image $\varphi(U)$ of any neighborhood of zero $U \subseteq X$ is a neighborhood of zero in the subspace $\varphi(X)$ of Y (with the topology inherited from Y):

$$\forall U \in \mathcal{U}(X) \quad \exists V \in \mathcal{U}(Y) \quad \varphi(U) \supseteq \varphi(X) \cap V.$$

Certainly, it is sufficient here to claim that U is open and absolutely convex. By the obvious formula

$$\varphi(X) \cap V = \varphi(\varphi^{-1}(V)), \quad V \subseteq Y, \quad (3.1)$$

(valid for any map of sets $\varphi : X \rightarrow Y$ and for any subset $V \subseteq Y$), this condition can be rewritten as follows:

$$\forall U \in \mathcal{U}(X) \quad \exists V \in \mathcal{U}(Y) \quad \varphi(U) \supseteq \varphi(\varphi^{-1}(V)).$$

In the stereotype theory the following property is dual.

- Let us say that a linear continuous map of stereotype spaces $\varphi : X \rightarrow Y$ is *closed*, if it is a covering onto its closed image, i.e. for any compact set $S \subseteq \overline{\varphi(X)} \subseteq Y$ there is a compact set $T \subseteq X$ such that $S \subseteq \varphi(T)$. Certainly, this means in particular that the set of values $\varphi(X)$ of φ must be closed in Y .

Theorem 3.2. *For a morphism of stereotype spaces $\varphi : X \rightarrow Y$ the following conditions are equivalent:*

- (i) *the map $\varphi : X \rightarrow Y$ is open;*
- (ii) *the map $\varphi^* : Y^* \rightarrow X^*$ is closed.*

Proof. 1. (i) \implies (ii). Let $\varphi : X \rightarrow Y$ be open. Take a compact set $S \in \mathcal{BK}(\overline{\varphi^*(Y^*)})$, i.e. $S \in \mathcal{BK}(X^*)$ and $S \subseteq \varphi^*(Y^*)$. Its polar $U = {}^\circ S$ is a neighborhood of zero in X , hence from the openness of φ it follows that there exists a neighborhood of zero $V \in \mathcal{BU}(Y)$ such that $\varphi(U) \supseteq \varphi(X) \cap V$. Its polar $T = V^\circ$ must be an absolutely convex compact in Y . Let us show that $S \subseteq \varphi^*(T)$.

We have to take an arbitrary functional $f \in S$ and to show that there exists $g \in T$ such that $f = \varphi^*(g)$. Since $S \subseteq \overline{\varphi^*(Y^*)} = [2, (3.2)] = \varphi^{-1}(0)^\perp$, we have that $\varphi^{-1}(0) \subseteq f^{-1}(0)$. Thus f can be represented as a composition

$$f = h \circ \varphi,$$

where h is a (uniquely defined) functional on $\varphi(X)$ (its continuity must be checked). We have:

$$1 \geq \sup_{x \in U} |f(x)| = \sup_{x \in U} |h(\varphi(x))| = \sup_{y \in \varphi(U)} |h(y)| \geq \sup_{y \in \varphi(X) \cap V} |h(y)|.$$

That is, the functional h is bounded by identity on the intersection of the unit ball V of the seminorm $p(y) = \inf\{\lambda > 0 : y \in \lambda \cdot V\} = \sup_{g \in T} |g(y)|$ with the subspace $\varphi(X)$, where h is defined. Or, in other words, the functional h is subordinated to the seminorm p on the subspace $\varphi(X)$. By the Hahn-Banach theorem this means that h can be extended to a linear continuous functional $g \in Y^*$, which is also subordinated to the seminorm p , and, as a corollary, g lies in $V^\circ = T$:

$$h = g|_{\varphi(X)}, \quad g \in T.$$

Since on the set $\varphi(X)$ the functionals h and g coincide, we have

$$f = h \circ \varphi = g \circ \varphi = \varphi^*(g), \quad g \in T.$$

2. (ii) \implies (i). On the contrary, suppose $\varphi^* : Y^* \rightarrow X^*$ is closed. Take an open and absolutely convex neighborhood of zero U in X . The set $U + \varphi^{-1}(0)$ is also a neighborhood of zero in X , hence its polar $S = (U + \varphi^{-1}(0))^\circ$ is compact in X^* . From the condition

$$\varphi^{-1}(0) \subseteq U + \varphi^{-1}(0)$$

it follows that S is contained in $\overline{\varphi^*(Y^*)}$:

$$f \in S = (U + \varphi^{-1}(0))^\circ \implies f \in \varphi^{-1}(0)^\perp = [2, (3.2)] = \overline{\varphi^*(Y^*)}$$

Since φ^* is a covering onto its image $\overline{\varphi^*(Y^*)}$, there is a compact set $T \subseteq Y^*$ such that

$$S = \varphi^*(T).$$

For the neighborhood of zero $V = \{y \in Y : \sup_{g \in T} |g(y)| < 1\}$ in the space Y we have:

$$\begin{aligned} y \in \varphi(U) &\iff y \in \varphi(U + \varphi^{-1}(0)) \iff \exists x \in \underbrace{U + \varphi^{-1}(0)}_{\parallel \{x \in X : \sup_{f \in S} |f(x)| < 1\}} \quad y = \varphi(x) \iff \\ &\iff \exists x \in X \quad y = \varphi(x) \ \& \ \sup_{f \in S} |f(x)| < 1 \iff \exists x \in X \quad y = \varphi(x) \ \& \ \sup_{f \in \varphi^*(T)} |f(x)| < 1 \iff \\ &\iff \exists x \in X \quad y = \varphi(x) \ \& \ \sup_{g \in T} |\varphi^*(g)(x)| = |g(\varphi(x))| < 1 \iff \\ &\iff \exists x \in X \quad y = \varphi(x) \ \& \ \sup_{g \in T} |g(y)| < 1 \iff y \in \varphi(X) \ \& \ y \in V \iff y \in \varphi(X) \cap V \end{aligned}$$

That is $\varphi(U) = \varphi(X) \cap V$, and thus, φ must be open. \square

Weakly open and weakly closed morphisms. The following two conditions are weakening of openness and closure.

- Let us say that a linear continuous map of stereotype spaces $\varphi : X \rightarrow Y$ is
 - *weakly open*, if the image $\varphi(U)$ of each X^* -weak neighborhood of zero $U \subseteq X$ is a Y^* -weak neighborhood of zero in the subspace $\varphi(X)$ of Y (with the topology inherited from Y):

$$\forall U \in \mathcal{U}(X_w) \quad \exists V \in \mathcal{U}(Y_w) \quad \varphi(U) \supseteq \varphi(X) \cap V$$

(here X_w denote the space X with the X^* -weak topology, and the same for Y_w); certainly, it is sufficient here to claim that U is X^* -weakly open and absolutely convex, and V is arbitrary (not necessarily Y^* -weak) neighborhood of zero in Y ;

- *weakly closed*, if it is a surjection onto its closed image; equivalently, the set $\varphi(X)$ of values of the map φ must be closed in Y :

$$\overline{\varphi(X)}^Y = \varphi(X).$$

Proposition 3.1. *For a morphism of stereotype spaces $\varphi : X \rightarrow Y$*

- if φ is open, then φ is weakly open,
- if φ is closed, then φ is weakly closed.

Proof. The second part of this proposition is obvious, and we already noticed this when defining closed morphisms. So the first part is of interest here. Note, that it is sufficient to assume that X and Y are just locally convex spaces. Suppose $\varphi : X \rightarrow Y$ is an open map, and let us prove that it is weakly open. Let us assume first that $\varphi : X \rightarrow Y$ is surjective: $\varphi(X) = Y$. Take a X^* -weak and absolutely convex neighborhood of zero U in X , i.e. $U \in \mathcal{U}(X)$ and its kernel $\text{Ker } U = \bigcap_{\lambda > 0} \lambda \cdot U$ has finite codimension in X . Then if φ is open, then the image $\varphi(U)$ of U is a neighborhood of zero in Y , and its kernel $\text{Ker } \varphi(U) = \bigcap_{\lambda > 0} \lambda \cdot \varphi(U) = \varphi(\text{Ker } U)$ has finite dimension in Y (since a surjective operator turns a subspace of finite codimension into a subspace of finite codimension). Hence, $\varphi(U)$ is a Y^* -weak neighborhood of zero in Y .

Let now $\varphi(X) \neq Y$. If φ is open, then as we already understood the image $\varphi(U)$ of any X^* -weak absolutely convex neighborhood of zero U in X is a $\varphi(X)^*$ -weak neighborhood of zero in $\varphi(X)$ (of course, we endow $\varphi(X)$ with the topology inherited from Y). By the Hahn-Banach theorem each functional $f \in \varphi(X)^*$ can be extended to a functional $g \in Y^*$, so the weak topology on $\varphi(X)$ generated by functionals from $\varphi(X)^*$ is the same as the weak topology generated by functionals from Y^* . \square

Theorem 3.3. *For a morphism of stereotype spaces $\varphi : X \rightarrow Y$ the following conditions are equivalent:*

- (i) *the map φ is weakly open;*
- (ii) *the map $\varphi^* : Y^* \rightarrow X^*$ is weakly closed.*

Proof. 1. Suppose φ is weakly closed. Take a functional $f \in \overline{\varphi^*(Y^*)} = [2, (3.2)] = \varphi^{-1}(0)^\perp$. Since $\varphi^{-1}(0) \subseteq f^{-1}(0)$, we can represent this functional as a composition

$$f = h \circ \varphi,$$

where h is a (uniquely defined) linear functional on $\varphi(X)$ (its continuity requires verification). Note that φ turns the polar $U = \{x \in X : |f(x)| \leq 1\} = {}^\circ f$ of the functional f into the polar of the functional h :

$$\varphi(U) = \{y \in \varphi(X) : |h(y)| \leq 1\} = {}^\circ h.$$

Indeed,

$$\begin{aligned} y \in \varphi(U) &\iff \exists x \in U \quad y = \varphi(x) \iff \exists x \in X \quad y = \varphi(x) \text{ \& } |f(x)| \leq 1 \iff \\ &\iff \exists x \in X \quad y = \varphi(x) \text{ \& } |h(\varphi(x))| \leq 1 \iff \\ &\iff \exists x \in X \quad y = \varphi(x) \text{ \& } |h(y)| \leq 1 \iff y \in \varphi(X) \text{ \& } |h(y)| \leq 1 \end{aligned}$$

Since the functional f is continuous on X , its polar $U = \{x \in X : |f(x)| \leq 1\} = {}^\circ f$ is an X^* -weak neighborhood of zero in X . On the other hand, φ is weakly open, hence there exists an Y^* -weak neighborhood of zero V in Y such that

$${}^\circ h = \varphi(U) \supseteq \varphi(X) \cap V.$$

Thus, the functional h is bounded on some neighborhood of zero $\varphi(X) \cap V$ in $\varphi(X)$. This means that it is continuous on $\varphi(X)$, and therefore we can extend it by the Hahn-Banach theorem to some continuous linear functional g on Y . The functionals g and h coincide on the set $\varphi(X)$, so we have

$$f = h \circ \varphi = g \circ \varphi = \varphi^*(g), \quad g \in Y^*.$$

This proves the closure of $\varphi^*(Y^*)$ in X^* .

2. Conversely, suppose that $\varphi^*(Y^*)$ is closed in X^* . Let us first take a functional $f \in X^*$ and show that its polar $U = \{x \in X : |f(x)| \leq 1\} = {}^\circ f$ is turned by the map φ into some neighborhood of zero in $\varphi(X)$. Note that if $\varphi^{-1}(0) \not\subseteq f^{-1}(0)$, then $\varphi(U) = \varphi(X)$, and the situation becomes trivial. So only the case when $\varphi^{-1}(0) \subseteq f^{-1}(0)$ is of interest. We can rewrite this inclusion as follows:

$$f \in \left(\varphi^{-1}(0)\right)^{\perp \Delta} = [2, (3.2)] = \overline{\varphi^*(Y^*)} = \left(\overline{\varphi^*(Y^*)}\right)^\Delta = \left(\varphi^*(Y^*)\right)^\Delta$$

(the last equality follows from the fact that $\varphi^*(Y^*)$ is closed in X^*). In particular this means that there exists a functional $h \in Y^*$ such that

$$f = \varphi^*(h) = h \circ \varphi.$$

The same reasoning as in the first part of this proof shows that the polar $V = \{y \in Y : |h| \leq 1\} = {}^\circ h$ has the following property:

$$\varphi(U) = \varphi(X) \cap V.$$

Let now U be an arbitrary X^* -weak absolutely convex neighborhood of zero in X . The set $\tilde{U} = U + \varphi^{-1}(0)$ is also an X^* -weak absolutely convex neighborhood of zero in X , and in addition

$$\varphi(U) = \varphi(\tilde{U}), \quad \tilde{U} + \varphi^{-1}(0) = \tilde{U}.$$

The second equality here implies that \tilde{U} contains a polar ${}^\circ\{f_1, \dots, f_k\}$ of some finite sequence of functionals $f_i \in X^*$ such that $\varphi^{-1}(0) \subseteq f_i^{-1}(0)$. Put $U_i = {}^\circ f_i$, then as we already proved, there exists a Y^* -weak neighborhood of zero V_i in Y such that $\varphi(U_i) = \varphi(X) \cap V_i$. So we can set $V = V_1 \cap \dots \cap V_k$, and this will be an Y^* -weak neighborhood of zero in Y , and we obtain

$$\varphi(U) = \varphi(\tilde{U}) \supseteq \varphi(U_1 \cap \dots \cap U_k) = \varphi(V_1) \cap \dots \cap \varphi(V_k) = \varphi(X) \cap V_1 \cap \dots \cap V_k = \varphi(X) \cap V.$$

□

- If a map $\varphi : X \rightarrow Y$ is injective and weakly open, then we call it a *weak embedding*.

From Theorem 3.3 we have

Corollary 3.1. *A morphism of stereotype spaces $\mu : X \rightarrow Y$ is a weak embedding \iff its dual morphism $\mu^* : Y^* \rightarrow X^*$ is a surjective map.*

Relatively open and relatively closed morphisms. Another weakening of openness and closure of morphisms is the following.

- We say that a linear continuous map of stereotype spaces $\varphi : X \rightarrow Y$ is
 - *relatively open*, if for each neighborhood of zero U in X (without loss of generality we may assume that U is closed and absolutely convex) such that every functional $f \in X^*$ bounded on U can be extended along the map φ to some functional $g \in Y^*$,

$$\forall f \in X^* \quad \left(\sup_{x \in U} |f(x)| < \infty \implies \exists g \in Y^* \quad f = g \circ \varphi \right), \quad (3.2)$$

its image $\varphi(U)$ is a neighborhood of zero in the subspace $\varphi(X)$ of the locally convex space Y (with the topology inherited from Y):

$$\varphi(U) \supseteq V \cap \varphi(X)$$

for some neighborhood of zero V in Y ;

- *relatively closed*, if for each absolutely convex compact set $T \subseteq Y$, if T contains in $\varphi(X)$, then there is a compact set $S \subseteq X$ such that $T \subseteq \varphi(S)$.

The following is obvious:

Proposition 3.2. *For a morphism of stereotype spaces $\varphi : X \rightarrow Y$*

- *if φ is open, then φ is relatively open,*
- *if φ is closed, then φ is relatively closed.*

Theorem 3.4. *For a morphism of stereotype spaces $\varphi : X \rightarrow Y$ the following conditions are equivalent:*

- (i) *the map φ is relatively open;*
- (ii) *the dual map $\varphi^* : Y^* \rightarrow X^*$ is relatively closed.*

Proof. 1. (i) \implies (ii). Let φ be relatively open, and T be an absolutely convex compact set in X^* , which in addition is contained in $\varphi^*(Y^*)$, i.e.

$$\forall f \in T \quad \exists g \in Y^* \quad f = \varphi^*(g) = g \circ \varphi. \quad (3.3)$$

For the polar $U = {}^\circ T$ this means that the condition (3.2) holds, and, since U is a neighborhood of zero in X , the image $\varphi(U)$ must be a neighborhood of zero in the subspace $\varphi(X)$ of the locally convex space Y (with the topology inherited from Y). Thus, there exists an absolutely convex neighborhood of zero V in Y such that

$$\varphi(U) \supseteq V \cap \varphi(X).$$

Let us put $S = V^\circ$ and show that $T \subseteq \varphi^*(S)$, i.e.

$$\forall f \in U^\circ \quad \exists h \in V^\circ \quad f = \varphi^*(h) = h \circ \varphi. \quad (3.4)$$

Indeed, take $f \in T = U^\circ$. Then by (3.3) we can find $g \in Y^*$ such that $f = g \circ \varphi$. The restriction $g|_{\varphi(X)}$ of this functional g on the subspace $\varphi(X)$ is bounded on the neighborhood of zero $V \cap \varphi(X)$:

$$\sup_{y \in V \cap \varphi(X)} |g(y)| \leq \sup_{y \in \varphi(U)} |g(y)| \leq \sup_{x \in U} |g(\varphi(x))| = \sup_{x \in U} |f(x)| \leq 1.$$

This means that the functional $g|_{\varphi(X)}$ is subordinated (on the subspace $\varphi(X)$) to the seminorm

$$p(y) = \inf\{\lambda > 0 : y \in \lambda \cdot V\}.$$

By the Hahn-Banach theorem we can extend $g|_{\varphi(X)}$ to some functional h on Y subordinated to p :

$$\begin{cases} |h(y)| \leq p(y), & y \in Y \\ h(y) = g(y), & y \in \varphi(X). \end{cases}$$

From the first condition here we can conclude that $\sup_{y \in V} |h(y)| \leq \sup_{y \in V} p(y) \leq 1$, i.e. $h \in V^\circ = S$. And from the second, that $h(\varphi(x)) = g(\varphi(x)) = f(x)$. Together this means (3.4).

2. (i) \Leftarrow (ii). Conversely, suppose $\varphi^* : Y^* \rightarrow X^*$ is relatively closed, and U is a closed absolutely convex neighborhood of zero in X satisfying (3.2). Its polar $T = U^\circ$ is an absolutely convex compact in X^* , and the condition (3.2) is equivalent to the condition (3.3). This in its turn means the inclusion $T \subseteq \varphi^*(Y^*)$, and since φ^* is relatively closed, there exists an absolutely convex compact set $S \subseteq Y^*$ such that

$$T \subseteq \varphi^*(S).$$

Therefore,

$$T^\circ \supseteq \left(\varphi^*(S)\right)^\circ = (\varphi^{**})^{-1}(S^\circ)$$

(in the equality we use the standard formula, see. e.g. [34, 2.3] or [2, (3.1)]). Passing from X^{**} and Y^{**} to X and Y , we have

$${}^\circ T \supseteq \varphi^{-1}({}^\circ S).$$

We can take now a neighborhood of zero $V = {}^\circ S$ in Y , and we obtain:

$$U \supseteq \varphi^{-1}(V) \implies \varphi(U) \supseteq \varphi(\varphi^{-1}(V)) = (3.1) = \varphi(X) \cap V.$$

That is what we need. □

- Let us call a morphism of stereotype spaces $\varphi : X \rightarrow Y$
 - a *relative embedding*, if it is injective and relatively open,
 - a *relative covering*, if it is relatively closed and $\varphi(X)$ is dense in Y .

Theorem 3.4 implies

Corollary 3.2. *A morphism of stereotype spaces $\mu : X \rightarrow Y$ is a relative embedding \iff its dual morphism $\mu^* : Y^* \rightarrow X^*$ is a relative covering.*

Connections between the three variations of openness and closure. Propositions 3.1 and 3.2 can be strengthened as follows.

Theorem 3.5. *For a morphism of stereotype spaces $\varphi : X \rightarrow Y$*

- (a) φ is open $\iff \varphi$ is weakly open and relatively open;
- (b) φ is closed $\iff \varphi$ is weakly closed and relatively closed.

Proof. Certainly, (a) and (b) are dual to each other, so it is sufficient to prove (b). The implication \Rightarrow is already sated in propositions 3.1 and 3.2, so we need to consider the inverse implication. If φ is weakly closed and relatively closed, then the first property means that $\overline{\varphi(X)} = \varphi(X)$, and the second, that every absolutely convex compact set $T \subseteq \varphi(X)$ is an image of some compact set $S \subseteq X$ under the map φ . These statements together mean that T can be chosen in $\overline{\varphi(X)}$, and the same will be true for it. This means that φ is closed. □

(b) Subspaces

- Let Y be a subset in a stereotype space X endowed with the structure of stereotype space in such a way that the set-theoretic enclosure $Y \subseteq X$ becomes a morphism of stereotype spaces (i.e. a linear continuous map). Then the stereotype space Y is called a *subspace* of the stereotype space X , and the set-theoretic enclosure $\sigma : Y \subseteq X$ its *representing monomorphism*. The record

$$Y \subsetneq X$$

or

$$X \supsetneq Y$$

will mean that Y is a subspace of the stereotype space X . If in addition we write

$$Y = X$$

then this means that the stereotype spaces Y and X coincide not only as sets but also with their algebraic and topological structure.

- The system of subspaces of a stereotype space X will be denoted by the symbol $\text{Sub}(X)$.

Proposition 3.3. *For a morphism $\mu : Z \rightarrow X$ in the category **Ste** of stereotype spaces the following conditions are equivalent:*

- (i) μ is a monomorphism,
- (ii) there exists a subspace Y in X with the representing monomorphism $\sigma : Y \subsetneq X$ and an isomorphism $\theta : Z \rightarrow Y$ of stereotype spaces such that the following diagram is commutative:

$$\begin{array}{ccc} Z & \xrightarrow{\mu} & X \\ \theta \downarrow \wr & & \uparrow \wr \sigma \\ Y & \xrightarrow{\sigma} & X \end{array}$$

Corollary 3.3. *For any stereotype space X the system $\text{Sub}(X)$ of its subspaces is a system of subobjects in X (in the sense of definition on page 9).*

Certainly, for a stereotype space P the relation \subsetneq is a partial order on the set $\text{Sub}(P)$ of subspaces of P .

Immediate subspaces.

- Suppose we have a sequence of two subspaces

$$Z \subsetneq Y \subsetneq X,$$

and the enclosure $Z \subsetneq Y$ is a bimorphism of stereotype spaces, i.e. apart from the other requirements, Z is dense in Y (with respect to the topology of Y):

$$\overline{Z}^Y = Y.$$

Then we will say that the subspace Y is a *mediator* for the subspace Z in the space X .

- We call a subspace Z of a stereotype space X an *immediate subspace* in X , if it has no non-isomorphic mediators, i.e. for any mediator Y in X the corresponding enclosure $Z \subsetneq Y$ is an isomorphism. In this case we use the record $Z \subsetneq X$:

$$Z \subsetneq X \iff \forall Y \left(\left(Z \subsetneq Y \subsetneq X \ \& \ \overline{Z}^Y = Y \right) \implies Z = Y \right).$$

Remark 3.1. In the category of locally convex spaces **LCS** the same construction gives a widely used object: immediate subspaces in a locally convex space X are exactly closed subspaces in X with the topology inherited from X . Below in Examples 3.2 and 3.3 we will see that in the category **Ste** of stereotype spaces the situation becomes sufficiently more complicated.

Recall that immediate monomorphisms were defined on page 13.

Proposition 3.4. *For a morphism $\mu : Z \rightarrow X$ in the category **Ste** the following conditions are equivalent:*

- (i) μ is an immediate monomorphism,
- (ii) there exists an immediate subspace Y of X with a representing monomorphism $\sigma : Y \hookrightarrow X$ and an isomorphism $\theta : Z \rightarrow Y$ such that the following diagram is commutative

$$\begin{array}{ccc}
 Z & \xrightarrow{\mu} & X \\
 \theta \downarrow \wr & & \uparrow \sigma \\
 Y & &
 \end{array}
 \quad (3.5)$$

The subspaces Y and the morphism θ here are uniquely defined by Z and μ .

Proof. The implication (i) \iff (ii) is obvious, so we need to prove only (i) \implies (ii). Put $Y = \mu(Z)$, and denote by $\theta : Z \rightarrow Y$ the co-restriction of μ on Y , i.e. θ is the same map as μ but it is assumed that θ acts from Z into Y . Since μ is injective, θ is bijective. Let us endow Y by the topology under which θ is an isomorphism of locally convex spaces. Then Y becomes a subspace of X , since for any neighborhood of zero U in X its inverse image $\mu^{-1}(U)$ must be a neighborhood of zero in Z , and thus the set $Y \cap U = \theta(\mu^{-1}(U))$ must be a neighborhood of zero in Y . \square

Proposition 3.5. ⁷ *For an immediate subspace Y of a stereotype space X with a representing monomorphism $\sigma : Y \subseteq X$ the following conditions are equivalent:*

- (i) σ is a closed map,
- (ii) σ is a weakly closed map,
- (iii) Y as a set is a closed subspace in the locally convex space X , and the topology of Y is a pseudosaturation of the topology inherited from X .

- If the conditions (i)-(iii) of this proposition are fulfilled, then we say that the immediate subspace Y of the space X is *closed*.

Proof. 1. The implication (i) \implies (ii) is a special case of the common situation stated in Proposition 3.1.

2. Let us prove (ii) \implies (iii). Let $\sigma : Y \subseteq X$ be a weakly closed map, i.e. Y as a set is closed in X . Denote by E the space Y with the topology inherited from X . Clearly, Y is continuously embedded into E , and, since Y is pseudosaturated, this enclosure preserves its continuity after passage from E to its pseudosaturation E^Δ (we use here the reasoning stated in Diagram [2, (1.26)]). Thus, we obtain a sequence of subspaces

$$Y \hookrightarrow E^\Delta \hookrightarrow X,$$

and, since Y and E^Δ coincide as sets, the first of these monomorphisms is a bimorphism. Hence, E^Δ is a mediator for Y , and we obtain that $Y = E^\Delta$.

3. The implication (iii) \implies (i) follows from the fact the pseudosaturation does not change the system of totally bounded subsets. \square

The following example is due to O. G. Smolyanov [31] and was mentioned in [2] (as Example 3.22). We will use it later as an important technical result:

Example 3.1. There exists a complete locally convex space E (and thus, E is a locally convex projective limit of Banach spaces) such that its dual space E^* is metrizable, but not complete. As a corollary, E is not pseudosaturated, and there exists a discontinuous linear functional $f : E \rightarrow \mathbb{C}$, which is continuous under the topology of pseudosaturation E^Δ .

⁷In [2] Theorem 4.14, which is equivalent to Proposition 3.5 here, and the more general Theorem 11.7, contain an inaccuracy: the requirement of closure of σ is omitted there.

Proof. This is the space $E = Y^\perp$ from Example [2, 3.22]. It is complete, since it is a closed subspace in the complete space $Z^* = \mathcal{D}^*(\mathbb{R})$. On the other hand, by Lemma of annihilator [2, Lemma 2.18], $E^* \cong X^*/E^\perp \cong Z/Y$, and the last space in this sequence is metrizable, but not complete. Thus,

$$E^* \neq (E^*)^\vee,$$

and we can extend this to the chain

$$E^* \neq (E^*)^\vee \cong [2, \text{Theorem 3.14}] \cong (E^\Delta)^*,$$

which means that there exists a functional $f \in (E^\Delta)^* \setminus E^*$. (It is sufficient here that E is pseudocomplete, while E^* is not pseudocomplete.) \square

Example 3.2. There exists a stereotype space P with a closed immediate subspace Q , which topology is not inherited from P , and, moreover, some continuous functionals $g \in Q^*$ cannot be continuously extended on P (in the formal language this means that the representing monomorphism $Q \hookrightarrow P$ is closed, but not a weakly open map).

Proof. Consider the space E from Example 3.1. It is complete, so it can be represented as a complete subspace in some stereotype space P with the topology inherited from P (for example, one can take as P the direct product of all Banach quotient spaces E/F of E). The space $Q = E^\Delta$ is the one with the required properties. Indeed, it is closed in P , since E is closed in P . On the other hand, the functional $f : Q \rightarrow \mathbb{C}$, described in Example 3.1, is continuous on $Q = E^\Delta$, but it cannot be continuously extended to P , since otherwise it would be continuous on E . \square

Example 3.3. There exists a stereotype space X with an immediate subspace Z , which is not closed as a subset in X . Hence the enclosure $Z \subseteq X$ is not a weakly closed morphism in the sense of definition on page 75 (in particular, the enclosure $Z \subseteq X$ is not isomorphic in $\mathbf{Mono}(X)$ to a kernel of some other morphism $\varphi : X \rightarrow A$ in \mathbf{Ste}).

Proof. Let E and f be the space and the functional from Example 3.1. Consider the kernel $F = \{x \in E^\Delta : f(x) = 0\}$ of f , and endow F with the topology inherited from E^Δ (as a locally convex space F is a closed subspace in E^Δ). By [2, Proposition 3.19], E^Δ is complete, hence F is also complete, and again by [2, Proposition 3.19], its pseudosaturation $Z = F^\Delta$ must be complete. In addition, $Z = F^\Delta$ is pseudosaturated, and thus, stereotype. Note then, that since E is complete, it can be represented (as a locally convex space) as a closed subspace in a direct product X of some Banach space (in such a way that the topology of E is inherited from X). We will show that Z is an immediate subspace, but not a closed set in X .

First let us show that Z is not closed in X . As a set Z coincides with F , which is dense in E (in the topology of E , which is inherited from X). Hence,

$$\overline{Z}^X = \overline{F}^X = E \neq F = Z$$

(here $\overline{}^X$ means closure in X , as we settled on page 3). Now let us show that Z is an immediate subspace in X . Let Y be a mediator of Z in X :

$$Z \subseteq Y \subseteq X$$

From the fact that Z is dense in Y we obtain the following chain

$$\begin{aligned} \overline{Z}^Y &= Y \\ \Downarrow \\ \overline{Y}^X &= \overline{\overline{Z}^Y}^X = \overline{Z}^X = E \\ \Downarrow \\ Y &\subseteq E. \end{aligned}$$

This is an enclosure of sets. Note now that since Y is a subspace in X , the topology of Y must majorate the topology inherited from X , or, what is the same, the topology inherited from E . That is why the enclosure $Y \subseteq E$ is continuous, and therefore Y is a subspace in E . This implies that the pseudosaturation of Y must be a subspace in the pseudosaturation of E , and, since Y is pseudosaturated, we obtain a continuous enclosure:

$$Y = Y^\vee \subseteq E^\vee.$$

Thus, Y is a subspace in E^∇ .

Let us now forget about X and consider the following chain of subspaces:

$$Z \subseteq Y \subseteq E^\nabla.$$

From the fact that Z is a dense subspace in Y we obtain a new logical chain:

$$\begin{aligned} \overline{Z}^Y &= Y \\ \Downarrow \\ \overline{Y}^{E^\nabla} &= \overline{\overline{Z}^Y}^{E^\nabla} = \overline{Z}^{E^\nabla} = F \\ \Downarrow \\ Y &\subseteq F. \end{aligned}$$

Again this is an enclosure of sets. Then we note that since Y is a subspace in E^∇ , the topology of Y must majorate the topology inherited from E^∇ , or, what is the same, the topology inherited from F . Thus the enclosure $Y \subseteq F$ is continuous, and, as a corollary, Y is a subspace in F . This implies that the pseudosaturation of Y must be a subspace in the pseudosaturation of F , and, since Y is pseudosaturated, we obtain a continuous enclosure:

$$Y = Y^\nabla \subseteq F^\nabla = Z.$$

Thus, Y is a subspace in $F^\nabla = Z$. On the other hand, from the very beginning Z was a subspace in Y . Hence, $Z = Y$. \square

Envelope $\text{Env}^X M$ of a set M of elements in a space X . Theorem 3.23 which we will prove later, justifies the following definition.

- The *envelope* of a set $M \subseteq X$ in a stereotype space X is a subspace in X , denoted by $\text{Env}^X M$ or by $\text{Env } M$, and defined as the projective limit in the category **Ste**

$$\text{Env}^X M = \text{Env } M = \text{Ste-}\varprojlim E_i \tag{3.6}$$

of a contravariant system $\{E_i; i \in \mathbf{Ord}\}$ of subspaces in X , indexed by ordinal numbers and defined by the following inductive rules:

- 0) the space E_0 is defined as the pseudosaturation of the closure of linear span $\text{Span } M$ of the set M in the space X :

$$E_0 = \left(\overline{\text{Span } M}^X \right)^\Delta$$

- 1) suppose that for some ordinal number $j \in \mathbf{Ord}$ all the spaces $\{E_i; i < j\}$ are already defined, then the space E_j is defined as follows:

- if j is an isolated ordinal number, i.e. $j = i + 1$ for some i , then $E_j = E_{i+1}$ is defined as the pseudosaturation of the closure of linear span $\text{Span } M$ of the set M in the space E_i :

$$E_j = E_{i+1} = \left(\overline{\text{Span } M}^{E_i} \right)^\Delta$$

- if j is a limit ordinal number, i.e. $j \neq i + 1$ for any i , then E_j is defined as the projective limit in the category **Ste** of the net $\{E_i; i \rightarrow j\}$:

$$E_j = \varprojlim_{i \leftarrow j} E_i,$$

- this means that as a set E_j is the intersection of the spaces $\{E_i; i \rightarrow j\}$,

$$E_j = \bigcap_{i < j} E_i,$$

and the topology in E_j is the weakest stereotype locally convex topology, under which all the enclosures $E_j \subseteq E_i$ are continuous.

Since the transfinite sequence $\{E_i; i \in \mathbf{Ord}\}$ cannot be an injective map from the class of all ordinal numbers \mathbf{Ord} to the set $\mathbf{Sub}(X)$ of all subspaces of a stereotype space X , it must stabilize, i.e. after some number $k \in \mathbf{Ord}$ all the spaces E_i must coincide (with their topologies):

$$\forall l \geq k \quad E_l = E_k. \quad (3.7)$$

This implies that the contravariant system $\{E_i; i \in \mathbf{Ord}\}$ indeed has a projective limit, and this is exactly the subspace E_k in X .

Example 3.4. If a set M is total in X , then its envelope coincides with X :

$$\overline{\text{Span } M}^X = X \implies \text{Env } M = X$$

Proof. The equality $\overline{\text{Span } M}^X = X$ implies $E_0 = \left(\overline{\text{Span } M}^X\right)^\Delta = X$, and after that all the spaces E_i become equal to X

$$X = E_0 = E_1 = \dots$$

Hence, $\text{Env } M = X$. \square

Example 3.5. If a set M forms a closed subspace in X (as in a locally convex space), then its envelope coincides with the pseudosaturation of M with respect to the topology inherited from X :

$$\overline{\text{Span } M}^X = M \implies \text{Env } M = M^\Delta$$

Proof. From $\overline{\text{Span } M}^X = M$ we have $E_0 = \left(\overline{\text{Span } M}^X\right)^\Delta = M^\Delta$, then $E_1 = \left(\overline{\text{Span } M}^{E_0}\right)^\Delta = M^\Delta = E_0$, and all the other spaces E_i coincide with E_0 . Thus, $\text{Env } M = E_0 = M^\Delta$. \square

Theorem 3.6. The envelope $\text{Env}^X M$ of each set $M \subseteq X$ is an immediate subspace in X , containing M as a total subset:

$$M \subseteq \text{Env}^X M \subsetneq X, \quad \overline{\text{Span } M}^{\text{Env}^X M} = \text{Env}^X M. \quad (3.8)$$

Proof. 1. First let us verify that M is total in $\text{Env}^X M$. Suppose k is an ordinal number after which the sequence $\{E_i; i \in \mathbf{Ord}\}$ is stabilized, i.e. (3.7) holds. Then $\text{Env } M = E_k$, and if it turned out that M is not total in E_k , then we would have a contradiction with (3.7):

$$E_{k+1} = \overline{\text{Span } M}^{E_k} \neq E_k.$$

2. Let us show that $\text{Env } M$ is an immediate subspace in X . Suppose Y is a subspace in X such that

$$\text{Env } M \subseteq Y \subseteq X,$$

and $\text{Env } M$ is dense in Y . Since, as we already understood, $\text{Span } M$ is dense in $\text{Env } M$, we have

$$Y = \overline{\text{Span } M}^Y. \quad (3.9)$$

Now by induction we have that Y is continuously embedded into each E_i :

0) for $i = 0$ we have a chain

$$Y \subsetneq X \implies Y = (3.9) = \overline{\text{Span } M}^Y \subsetneq \overline{\text{Span } M}^X \implies Y = Y^\Delta \subsetneq \left(\overline{\text{Span } M}^X\right)^\Delta = E_0.$$

1) suppose that we proved $Y \subsetneq E_i$ for all i less than some j , then

— if j is an isolated ordinal number, i.e. $j = i + 1$ for some i , then

$$Y \subsetneq E_i \implies Y = (3.9) = \overline{\text{Span } M}^Y \subsetneq \overline{\text{Span } M}^{E_i} \implies Y = Y^\Delta \subseteq \left(\overline{\text{Span } M}^{E_i}\right)^\Delta = E_{i+1} = E_j,$$

— if j is a limit ordinal number, then from the continuous enclosures $Y \subsetneq E_i$ for $i < j$ we obtain a continuous enclosure of locally convex spaces

$$Y \subsetneq \text{LCS-}\lim_{j \leftarrow i} E_i,$$

and this implies a continuous enclosure of stereotype spaces

$$Y = Y^\Delta \subsetneq \left(\text{LCS-}\lim_{j \leftarrow i} E_i\right)^\Delta = \text{Ste-}\lim_{j \leftarrow i} E_i = E_j.$$

From the fact that Y is continuously embedded into each E_i we obtain a continuous enclosure $Y \subset \text{Env } M$. Together with the initial enclosure $\text{Env } M \subset Y$ this means the equality $\text{Env } M = Y$ (with topologies). \square

The following theorem shows that in an immediate subspace the topology is automatically defined by the set of its elements:

Theorem 3.7. *Every subspace Y in a stereotype space X is a subspace in its envelope $\text{Env}^X Y$*

$$Y \subset X \implies Y \subset \text{Env}^X Y, \quad (3.10)$$

and Y is an immediate subspace in X iff it coincide (with the topologies) with its envelope in X :

$$Y \subset X \iff Y = \text{Env}^X Y. \quad (3.11)$$

Proof. The continuity of the enclosure $Y \subset \text{Env}^X Y$ is proved by induction:

0) at the zero step we have a continuous enclosure of locally convex spaces

$$Y \subset \overline{\text{Span } Y}^X = \overline{Y}^X,$$

which implies a continuous enclosure of stereotype spaces

$$Y = Y^\Delta \subset \left(\overline{Y}^X \right)^\Delta = E_0,$$

1) suppose that the continuous enclosure $Y \subseteq E_i$ is proved for all i less than some j , then

— if j is an isolated ordinal number, i.e. $j = i + 1$ for some i , then we obtain a continuous enclosure of locally convex spaces

$$Y \subseteq \overline{\text{Span } Y}^{E_i} = \overline{Y}^{E_i},$$

which implies a continuous enclosure of stereotype spaces

$$Y = Y^\Delta \subseteq \left(\overline{Y}^{E_i} \right)^\Delta = E_{i+1} = E_j,$$

— if j is a limit ordinal number, then from the continuous enclosures $Y \subseteq E_i$ for all $i < j$ we obtain a continuous enclosure of locally convex spaces

$$Y \subseteq \text{LCS-lim}_{j \leftarrow i} E_i,$$

which implies a continuous enclosure of stereotype spaces

$$Y = Y^\Delta \subseteq \left(\text{LCS-lim}_{j \leftarrow i} E_i \right)^\Delta = \text{Ste-lim}_{j \leftarrow i} E_i = E_j.$$

Let us now consider a special case when Y is an immediate subspace in X . Then by Theorem 3.6, Y is dense in $\text{Env } Y$, hence in the chain of enclosures

$$Y \subseteq \text{Env } Y \subseteq X$$

the second space is a mediator. Therefore, it coincides with the first one: $Y = \text{Env } Y$. \square

Corollary 3.4. *The representing monomorphism $\sigma : Y \subset X$ of an immediate subspace Y in a stereotype space X is always relatively closed.*

Proof. This follows from the fact that in the chain of spaces $\{E_i\}$ (which defines $\text{Env}^X Y = Y$) the passage from the bigger space E_i to the smaller one E_{i+1} the system of compact sets in E_{i+1} is inherited from E_i . \square

Theorem 3.8. *If $\varphi : Y \rightarrow X$ is a morphism of stereotype spaces, turning a set $N \subseteq Y$ into a set $M \subseteq X$,*

$$\varphi(N) \subseteq M,$$

then φ continuously maps $\text{Env}^Y N$ into $\text{Env}^X M$:

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ \uparrow & & \uparrow \\ \text{Env}^Y N & \longrightarrow & \text{Env}^X M \end{array}$$

In the special cases:

$$\left\{ \begin{array}{l} Y \sqsubset X \\ \cup \quad \cup \\ N \subseteq M \end{array} \right\} \implies \text{Env}^Y N \sqsubset \text{Env}^X M, \quad (3.12)$$

$$\left\{ \begin{array}{l} Y \sqsubset\!\!\!\sqsubset X \\ \cup \quad \cup \\ N \subseteq M \end{array} \right\} \implies \text{Env}^Y N \sqsubset\!\!\!\sqsubset \text{Env}^X M, \quad (3.13)$$

$$\left\{ \begin{array}{l} Y \sqsubset\!\!\!\sqsubset X \\ \cup \quad \cup \\ N = M \end{array} \right\} \implies \text{Env}^Y M = \text{Env}^X M. \quad (3.14)$$

Proof. Take a morphism $\varphi : Y \rightarrow X$ of stereotype spaces turning a set $N \subseteq Y$ into a set $M \subseteq X$, $\varphi(N) \subseteq M$. If by $\{F_i; i \in \text{Ord}\}$ and $\{E_i; i \in \text{Ord}\}$ we denote the sequences of subspaces in Y and X , which define $\text{Env} N$ and $\text{Env} M$ respectively,

$$\text{Env} N = \varprojlim F_i, \quad \text{Env} M = \varprojlim E_i,$$

then we can prove by induction, that for each i the map φ continuously maps F_i into E_i ,

$$\varphi : F_i \rightarrow E_i,$$

and this implies that φ continuously maps $\text{Env} N$ into $\text{Env} M$,

$$\varphi : \text{Env} N \rightarrow \text{Env} M.$$

Let us now observe the special cases.

1. If $N \subseteq M$ and $Y \sqsubset X$, then we consider the sequences $\{F_i; i \in \text{Ord}\}$ and $\{E_i; i \in \text{Ord}\}$ of subspaces in X , which define $\text{Env}^X N$ and $\text{Env}^X M$. By induction we obtain an enclosure of subspaces $F_i \sqsubset E_i$ for each i , and this gives the enclosure $\text{Env}^X N \sqsubset \text{Env}^X M$.

2. Suppose that $N \subseteq M$ and $Y \sqsubset\!\!\!\sqsubset X$. Then, by implication (3.12) we already proved, $\text{Env}^Y N \sqsubset \text{Env}^X M$. Let us show that in this enclosure $\text{Env}^Y N$ is an immediate subspace in $\text{Env}^X M$. Let Z be a mediator for $\text{Env}^Y N$ in $\text{Env}^X M$:

$$\text{Env}^Y N \sqsubset Z \sqsubset \text{Env}^X M, \quad \overline{\text{Env}^Y N}^Z = Z.$$

Consider the envelope $\text{Env}^X(Y \cup Z)$ of the set $Y \cup Z$ in the space X . We can include it into a diagram (where all the arrows are theoretic set enclosures, which are continuous maps):

$$\begin{array}{ccccc} Y & \longrightarrow & \text{Env}^X(Y \cup Z) & \longrightarrow & X \\ \uparrow & & \uparrow & & \uparrow \\ \text{Env}^Y N & \longrightarrow & Z & \longrightarrow & \text{Env}^X M \end{array}$$

By Theorem 3.6, N is total in $\text{Env}^Y N$, which in its turn is total in Z (since Z is a mediator). Hence, N is total in Z . On the other hand, $N \subseteq Y$, hence Y is dense in Z (in the topology of Z , and thus in the topology of X as well). From this we have that Y is dense in the subset $Y \cup Z$ of the space X , and again by Theorem 3.6, Y is dense in $\text{Env}^X(Y \cup Z)$.

This means that $\text{Env}^X(Y \cup Z)$ is a mediator for Y in the space X :

$$Y \sqsubset \text{Env}^X(Y \cup Z) \sqsubset X, \quad \overline{Y}^{\text{Env}^X(Y \cup Z)} = \text{Env}^X(Y \cup Z).$$

The condition $Y \sqsubset\!\!\!\sqsubset X$ implies the equality of stereotype spaces $Y = \text{Env}^X(Y \cup Z)$. This in its turn implies that $Z \sqsubset Y$, i.e. Z is a mediator for $\text{Env}^Y N$ in Y :

$$\text{Env}^Y N \sqsubset Z \sqsubset Y, \quad \overline{\text{Env}^Y N}^Z = Z.$$

By Theorem 3.6, $\text{Env}^Y N$ is an immediate subspace in Y , so we obtain an equality of stereotype spaces $\text{Env}^Y N = Z$.

3. Suppose that $N = M \subseteq Y \sqsubset\!\!\!\sqsubset X$. Then by property (3.13) which we already proved,

$$\text{Env}^Y M \sqsubset\!\!\!\sqsubset \text{Env}^X M.$$

On the other hand by property (3.12) which has already been proved as well, the chain $M \subseteq Y \subsetneq X$ implies

$$\text{Env}^X M \subsetneq \text{Env}^X Y = (3.11) = Y$$

Together this gives a chain

$$\text{Env}^Y M \subsetneq \text{Env}^X M \subsetneq Y.$$

By Theorem 3.6, the set M is total in $\text{Env}^X M$, hence the space $\text{Env}^Y M$ is total in $\text{Env}^X M$. Thus, $\text{Env}^X M$ is a mediator in this chain, and we obtain the equality $\text{Env}^Y M = \text{Env}^X M$. \square

Theorem 3.9. *The envelope $\text{Env}^X M$ of any set $M \subseteq X$ is a minimal subspace among all the immediate subspaces in X , which contain M , and in each of those immediate subspaces $Y \subsetneq X$ the space $\text{Env}^X M$ is an immediate subspace:*

$$\forall Y \quad \left(M \subseteq Y \subsetneq X \implies \text{Env}^X M \subsetneq Y \right). \quad (3.15)$$

Proof.

$$\text{Env}^X M \stackrel{(3.14)}{=} \text{Env}^Y M \stackrel{(3.8)}{\subsetneq} Y.$$

\square

Proposition 3.6. *If $Y \subsetneq X$ and $Z \subsetneq X$, then the condition $Z \subseteq Y$ implies $Z \subsetneq Y$. In a special case, when $Y \subsetneq X$ and $Z \subsetneq X$, the condition $Z \subseteq Y$ implies $Z \subsetneq Y$.*

Proof. If $Y \subsetneq X$, $Z \subsetneq X$, $Z \subseteq Y$, then

$$Z \stackrel{(3.10)}{\subsetneq} \text{Env}^X Z \stackrel{(3.14)}{=} \text{Env}^Y Z \stackrel{(3.8)}{\subsetneq} Y.$$

If $Y \subsetneq X$, $Z \subsetneq X$, $Z \subseteq Y$, then

$$Z \stackrel{(3.11)}{=} \text{Env}^X Z \stackrel{(3.14)}{=} \text{Env}^Y Z \stackrel{(3.8)}{\subsetneq} Y.$$

\square

(c) Quotient spaces

- Let X be a stereotype space, and

- 1) in X as in a locally convex space we take a closed subspace E ,
- 2) on the quotient space X/E we consider an arbitrary locally convex topology τ , which is majorated by the natural quotient topology of X/E ,
- 3) in the completion $(X/E)^\nabla$ of the locally convex space X/E with the topology τ we take a subspace Y , which contains X/E and is a stereotype space with respect to the topology inherited from $(X/E)^\nabla$.

Then we call the stereotype space Y a *quotient space of the stereotype space X* , and the composition $v = \sigma \circ \pi$ of the quotient map $\pi : X \rightarrow X/E$ and the natural enclosure $\sigma : X/E \rightarrow Y$ is called the *representing epimorphism* of the quotient space Y . The record

$$Y \leftarrow X$$

or the record

$$X \twoheadrightarrow Y$$

will mean that Y is a quotient space of the stereotype space X . The class of all quotient spaces of X will be denoted by $\text{Quot}(X)$. From its construction it is clear that $\text{Quot}(X)$ is a set.

The following is evident:

Proposition 3.7. *For a morphism $\varepsilon : Z \leftarrow X$ in the category **Ste** the following conditions are equivalent:*

- (i) ε is an epimorphism,

- (ii) there is a quotient space Y of X with the representing epimorphism $v : Y \leftarrow X$, and an isomorphism $\theta : Z \leftarrow Y$ such that the following diagram is commutative:

$$\begin{array}{ccc} Z & & \\ \uparrow \theta & \nwarrow \varepsilon & \\ Y & \xleftarrow{v} & X \end{array} \quad (3.16)$$

Corollary 3.5. For a stereotype space X the system $\text{Quot}(X)$ of all its quotient spaces is a system of quotient objects for X .

The formalization of the idea of quotient object we have presented here has a qualitative shortcoming in comparison with the notion of subspace which we considered above: the problem is that the relation \leftarrow does not establish a partial order in the system $\text{Quot}(P)$ of quotient spaces of a stereotype space P . By the set-theoretic reasons no one of axioms of partial order (reflexivity, antisymmetry and transitivity) holds for \leftarrow . In particular, the first two axioms do not hold since the situation when $Y \leftarrow X$ and at the same time $Y = X$ is impossible. To explain this, let us agree for simplicity that we do not take into account the necessity to pass to a subspace in the completion which was stated in the step 3 of our definition – then $Y \leftarrow X$ (and $Y \neq \emptyset$) implies by the axiom of regularity [16, Appendix, Axiom VII] that there exists an element $y \in Y$ such that $y \cap Y = \emptyset$. But if in addition $Y = X$, then the element y , being a coset of X , i.e. a non-empty subset in X , must have non-empty intersection $y \cap Y = y \cap X = y \neq \emptyset$ with $X = Y$. As to the transitivity, in the situation when $Z \leftarrow Y$ and $Y \leftarrow X$ the elements of Z are non-empty sets of elements of Y , and each such element is a non-empty set of elements of X . From the point of view of set theory this is not the same as if elements of Z were sets of elements of X , so in this situation the relation $Z \leftarrow X$ is also impossible. This forces us to introduce a new binary relation.

- Suppose $Y \leftarrow X$ and $Z \leftarrow X$. We will say that the quotient space Y *subordinates* the quotient space Z , and we write in this situation $Z \leq Y$, if there exists a morphism $\varkappa : Y \rightarrow Z$ such that the following diagram is commutative:

$$\begin{array}{ccc} Y & & \\ \downarrow \varkappa & \swarrow v_Y & \\ Z & \xleftarrow{v_Z} & X \end{array} \quad (3.17)$$

(here v_Y and v_Z are representing epimorphisms for Y and Z). The morphism \varkappa , if exists, must be, first, unique, and, second, an epimorphism.

For any stereotype space P the relation \leq is a partial order on the set $\text{Quot}(P)$ of quotient spaces of P .

Immediate quotient spaces.

- Let Y and Z be two quotient spaces of X such that

$$Z \leq Y,$$

and the epimorphism $\varkappa : Z \leftarrow Y$ in diagram (3.17) is a monomorphism (and hence, a bimorphism) of stereotype spaces. Then we will say that the quotient space Y is a *mediator* for the quotient space Z of the space X . One can notice that in this case Y is a subset in Z , so we will write $Z \supseteq Y$.

- We call a quotient space Z of a stereotype space X an *immediate quotient space* in X , if it has no non-isomorphic mediators, i.e. for any its mediator Y in X the corresponding epimorphism $Z \leftarrow Y$ is an isomorphism. We write in this case $Z \leftarrow_o X$:

$$Z \leftarrow_o X \quad \Longleftrightarrow \quad \forall Y \quad \left((Z \leq Y \ \& \ Y \leftarrow X \ \& \ Z \supseteq Y) \implies Z = Y \right).$$

- Let us say that an immediate quotient space $Y \leftarrow_o X$ *strongly subordinates* an immediate quotient space $Z \leftarrow_o X$, and write $Z \leq^o Y$, if there exists a strong epimorphism $\varkappa : Y \rightarrow Z$ such that diagram (3.17) is commutative.

Remark 3.2. In the category of locally convex spaces \mathbf{LCS} the immediate quotient spaces of a locally convex space X are exactly quotient space of X by closed subspaces with the usual quotient topologies. Like in the case of subspaces, in the category \mathbf{Ste} of stereotype spaces the situation becomes more complicated (see below Examples 3.6 and 3.7).

Recall that the notion of immediate epimorphism was defined on page 13. The following statement is dual to Proposition 3.4, and can be proved by the dual reasoning:

Proposition 3.8. *For a morphism $\varepsilon : Z \leftarrow X$ in the category \mathbf{Ste} the following conditions are equivalent:*

- (i) ε is an immediate epimorphism,
- (ii) there exists an immediate quotient space Y of the stereotype space X with the representing morphism $v : Y \leftarrow X$ and an isomorphism $\theta : Z \leftarrow Y$ such that the following diagram is commutative:

$$\begin{array}{ccc} & Z & \\ & \uparrow \theta & \\ & Y & \\ & \nwarrow v & \\ & X & \\ & \nearrow \varepsilon & \\ & Z & \end{array} \quad (3.18)$$

The quotient space Y and the morphism θ are uniquely defined by Z and ε .

Proposition 3.9. ⁸ *For an immediate quotient space Y of a stereotype space X with the representing epimorphism $v : Y \leftarrow X$ the following conditions are equivalent:*

- (i) v is an open map,
- (ii) v is a weakly open map,
- (iii) Y is a pseudosaturation $(X/E)^\nabla$ of the quotient space X/E of the locally convex space X (with the usual quotient topology) by some closed locally convex subspace E :

$$Y = (X/E)^\nabla.$$

- If the conditions (i)-(iii) of this proposition are fulfilled, then we say that the immediate quotient space Y of X is *open*.

Proof. 1. The implication (i) \implies (ii) is a special case of the common situation described in Proposition 3.1.

2. Let us prove (ii) \implies (iii). Suppose the representing epimorphism $v : Y \leftarrow X$ is a weakly open map. Denote by E its kernel. By definition of stereotype quotient space, Y is a pseudocomplete locally convex subspace in the completion $(X/E)^\nabla$ of the locally convex space X/E under some topology τ which is majorated by the quotient topology X/E , and X/E lies in Y as set. Thus, we can represent v as a diagram

$$\begin{array}{ccc} X/E & \xleftarrow{\pi} & X \\ \downarrow \sigma & \nearrow v & \\ Y & & \end{array}$$

where $\pi : X \rightarrow X/E$ is the usual quotient map of locally convex spaces, and $\sigma : X/E \rightarrow Y$ is a natural bimorphism. Since Y is pseudocomplete, σ can be extended to some morphism σ^∇ on pseudocompletion $(X/E)^\nabla$ of the space X/E (we use here the reasoning stated in diagram [2, (1.13)]):

$$\begin{array}{ccccc} (X/E)^\nabla & \xleftarrow{\nabla_{X/E}} & X/E & \xleftarrow{\pi} & X \\ & \searrow \sigma^\nabla & \downarrow \sigma & \nearrow v & \\ & & Y & & \end{array}$$

Note that σ^∇ is not only epimorphism (this follows from the property of epimorphisms 3° on page 7, since the composition $v = \sigma^\nabla \circ \nabla_{X/E} \circ \pi$ is an epimorphism), but also a monomorphism. This is proved as follows. The

⁸In author's paper [2] Theorem 4.16, which is equivalent to Proposition 3.9 here, as well as the more general proposition, Theorem 11.9, contain an inaccuracy: the requirement of openness of v is omitted there.

fact that v is weakly open, implies that σ is weakly open as well. This means that every linear continuous functional on X/E can be extended along the map σ to a linear continuous functional on Y . In other words, the dual map $\sigma' : Y' \rightarrow X'$ is a surjection. This implies that the pseudosaturation σ^∇ must be an injection⁹.

As a result, we have a chain of epimorphisms

$$Y \xleftarrow{\sigma^\nabla} (X/E)^\nabla \xleftarrow{X/E \circ \pi} X,$$

where the first morphism σ^∇ is a bimorphism. Thus, $(X/E)^\nabla$ is a mediator for Y , and we obtain the equality $Y = (X/E)^\nabla$.

3. The implication (iii) \implies (i) follows from the fact that pseudocompletion does not change the topology. \square

The following example is dual to Example 3.2:

Example 3.6. There exists a stereotype space P with an immediate quotient space of the form $Y = (P/E)^\nabla$, which cannot be represented in the form $Y = P/F$ for a subspace $F \subseteq P$ (in formal language this means that the representing epimorphism $Y \leftarrow_\circ P$ is open, but not closed).

Proof. The space Z from example [2, 3.22] is such a space. It contains a closed subspace E such that the locally convex quotient space Z/E is metrizable, but not complete. As a corollary, in the stereotype sense the space $(Z/E)^\nabla$ is an immediate quotient space, but it cannot be represented in the form Z/F , since F is uniquely defined as the kernel of the map $Z \rightarrow Y$, and hence must coincide with E . \square

From Example 3.3 we have

Example 3.7. There exists a stereotype space P with an immediate quotient space Y such that the representing epimorphism $Y \leftarrow_\circ P$ is not weakly open (in the sense of definition on page 75). As a corollary, Y is not representable in the form $Y = (P/E)^\nabla$ for a subspace $E \subseteq P$ (and hence is not isomorphic in $\mathbf{Epi}(P)$ to a cokernel of some morphism $\varphi : A \rightarrow P$ in **Ste**).

Imprint $\mathbf{Imp}^X F$ of a set F of functionals on a space X . Theorem 3.24 which we will prove later justifies the following definition.

- Let F be a set of linear continuous functionals on a stereotype space X . The *imprint* of the set of functionals F on X is a quotient space of X , denoted by $\mathbf{Imp}^X F$, or by $\mathbf{Imp} F$, and defined as the injective limit in the category **Ste**

$$\mathbf{Imp}^X F = \mathbf{Imp} F = \mathbf{Ste}\text{-}\varinjlim E_i \quad (3.19)$$

of the covariant system $\{E_i; i \in \mathbf{Ord}\}$ of quotient spaces of X indexed by ordinal numbers and defined by the following inductive rules:

- 0) the space E_0 is the pseudocompletion of the quotient space $X/\mathbf{Ker} F$ (with the usual quotient topology) of X by the common kernel $\mathbf{Ker} F = \bigcap_{f \in F} \mathbf{Ker} f$ of functionals from F :

$$E_0 = (X/\mathbf{Ker} F)^\nabla,$$

- 1) if for an ordinal number $j \in \mathbf{Ord}$ all the spaces $\{E_i; i < j\}$ are already defined, then the space E_j is defined as follows:

- if j is an isolated ordinal, i.e. $j = i + 1$ for some i , then $E_j = E_{i+1}$ is defined as the pseudocompletion of the quotient space $E_i/\mathbf{Ker} F$ (with the usual quotient topology):

$$E_j = E_{i+1} = (E_i/\mathbf{Ker} F)^\nabla,$$

- if j is a limit ordinal, i.e. $j \neq i + 1$ for all i , then E_j is defined as the injective limit in the category **Ste** of stereotype spaces of the net $\{E_i; i \rightarrow j\}$:

$$E_j = \mathbf{Ste}\text{-}\varinjlim_{i \rightarrow j} E_i = \left(\mathbf{LCS}\text{-}\varinjlim_{i \rightarrow j} E_i \right)^\nabla.$$

⁹We use here the following obvious property of pseudocompletion: if $\varphi : X \rightarrow Y$ is a monomorphism of locally convex space, such that the dual map $\varphi' : X' \leftarrow Y'$ is a surjection, then its pseudocompletion $\varphi^\nabla : X^\nabla \rightarrow Y^\nabla$ is also a monomorphism of locally convex spaces.

Since the transfinite sequence $\{E_i; i \in \mathbf{Ord}\}$ cannot be an injective map from the class \mathbf{Ord} of all ordinal numbers into the set $\mathbf{Quot}(X)$ of quotient spaces of X , it must stabilize, i.e. after some number i all the spaces E_i must coincide together with the topology. As a corollary, the formula (3.19) uniquely defines some quotient space $\mathbf{Imp} F$ of X .

Example 3.8. If a set of functionals F separates elements of X (in other words, the common kernel $\mathbf{Ker} F$ of functionals from F is zero), then the imprint of F on X coincide with X :

$$\mathbf{Ker} F = 0 \implies \mathbf{Imp}^X F = X.$$

Proof. From $\mathbf{Ker} F = \{x \in X : \forall f \in F f(x) = 0\} = 0$ we have $E_0 = (X/\mathbf{Ker} F)^\nabla = X$. As a corollary, all the other spaces E_i coincide with X

$$X = E_0 = E_1 = \dots$$

Thus, $\mathbf{Imp} F = X$. □

Example 3.9. If a set of functionals F is a closed subspace in X^* (as in a locally convex space), then the imprint of F on X is the open immediate quotient space of X by the common kernel $\mathbf{Ker} F$, i.e. coincides with the pseudocompletion of the locally convex quotient space $X/\mathbf{Ker} F$ with the usual quotient topology:

$$\overline{\mathbf{Span} F}^{X^*} = F \implies \mathbf{Imp}^X F = (X/\mathbf{Ker} F)^\nabla$$

The following two theorems are dual to Theorems 3.6 and 3.7, and therefore do not require proof.

Theorem 3.10. *The imprint $\mathbf{Imp}^X F$ of any set of functionals $F \subseteq X^*$ on a stereotype space X is an immediate quotient space of X , to which functionals from F can be continuously extended:*

$$\mathbf{Imp}^X F \hookleftarrow_{\circ}^v X, \quad \forall f \in F \quad \exists g \in (\mathbf{Imp}^X F)^* : f = g \circ v. \quad (3.20)$$

Theorem 3.11. *Every quotient space Y of a stereotype space X is subordinated to the imprint $\mathbf{Imp}^X(Y^* \circ v)$ of the system of functionals $Y^* \circ v = \{g \circ v; g \in Y^*\}$ on the space X , where $v : Y \hookleftarrow X$ is the representing epimorphism of Y :*

$$v : Y \hookleftarrow X \implies Y \leq \mathbf{Imp}^X(Y^* \circ v), \quad (3.21)$$

and Y is an immediate quotient subspace of X , iff Y coincides (as a locally convex space) with this imprint:

$$v : Y \hookleftarrow_{\circ} X \iff Y = \mathbf{Imp}^X(Y^* \circ v). \quad (3.22)$$

Corollary 3.6. *The representing epimorphism $v : Y \hookleftarrow_{\circ} X$ of any continuous quotient space Y of a stereotype space X is always relatively open.*

The following theorem is dual to Theorem 3.8.

Theorem 3.12. *If $\varphi : Y \hookleftarrow X$ is a morphism of stereotype spaces, turning a set of functionals $G \subseteq Y^*$ into a set of functionals $F \subseteq X^*$,*

$$G \circ \varphi \subseteq F,$$

then there exists a unique morphism $\varepsilon : \mathbf{Imp}^Y G \hookleftarrow \mathbf{Imp}^X F$ such that the following diagram is commutative:

$$\begin{array}{ccc} Y & \xleftarrow{\varphi} & X \\ \downarrow & & \downarrow \\ \mathbf{Imp}^Y G & \xleftarrow{\varepsilon} & \mathbf{Imp}^X F \end{array}$$

In the special cases:

$$\left\{ \begin{array}{l} \varphi : Y \hookleftarrow X \\ G \circ \varphi \subseteq F \end{array} \right\} \implies \varepsilon \text{ is an epimorphism}, \quad (3.23)$$

$$\left\{ \begin{array}{l} \varphi : Y \hookleftarrow_{\circ} X \\ G \circ \varphi \subseteq F \end{array} \right\} \implies \varepsilon \text{ is an immediate epimorphism}, \quad (3.24)$$

$$\left\{ \begin{array}{l} \varphi : Y \hookleftarrow_{\circ} X \\ G \circ \varphi = F \end{array} \right\} \implies \varepsilon \text{ is an isomorphism}. \quad (3.25)$$

Theorem 3.13. *The imprint $\text{Imp}^X F$ of a set $F \subseteq X^*$ of functionals on a stereotype space X is a minimal quotient space among immediate quotient spaces of X to which functionals F can be extended. Moreover, every such quotient space Y strongly subordinates $\text{Imp}^X F$:*

$$\forall Y \quad \left(F \subseteq Y^* \ \& \ Y \leftarrow_{\circ} X \implies \text{Imp}^X F \leq^{\circ} Y \right). \quad (3.26)$$

Proposition 3.10. *If $\alpha : Y \leftarrow_{\circ} X$ and $\beta : Z \leftarrow_{\circ} X$, then the condition $Z^* \circ \alpha \subseteq Y^* \circ \beta$ implies $Z \leq Y$. In a special case, when $Y \subset_{\circ} X$ and $Z \subset_{\circ} X$, the condition $Z^* \circ \alpha \subseteq Y^* \circ \beta$ implies $Z \leq^{\circ} Y$.*

(d) Decompositions, factorizations, envelope and imprint in **Ste**.

Pre-abelian property and basic decomposition in **Ste.** Since any two parallel morphisms $X \begin{smallmatrix} \xrightarrow{\varphi} \\ \xrightarrow{\psi} \end{smallmatrix} Y$ in the category **Ste** of stereotype spaces can be added and subtracted one from another, it is clear that **Ste** is an additive category. In [2] it was noticed that this category is pre-abelian:

Theorem 3.14. *In the category **Ste** of stereotype spaces for each morphism $\varphi : X \rightarrow Y$ the formulas*

$$\text{Ker } \varphi = \left(\varphi^{-1}(0) \right)^{\Delta}, \quad \text{Coker } \varphi = \left(Y / \overline{\varphi(X)} \right)^{\nabla}, \quad \text{Coim } \varphi = \left(X / \varphi^{-1}(0) \right)^{\nabla}, \quad \text{Im } \varphi = \left(\overline{\varphi(X)} \right)^{\Delta} \quad (3.27)$$

define respectively kernel, cokernel, coimage and image. The operation $\varphi \mapsto \varphi^$ of taking dual map establishes the following connections between these objects:*

$$(\text{ker } \varphi)^* = \text{coker } \varphi^* \quad (\text{coker } \varphi)^* = \text{ker } \varphi^* \quad (\text{im } \varphi)^* = \text{coim } \varphi^* \quad (\text{coim } \varphi)^* = \text{im } \varphi^* \quad (3.28)$$

$$(\text{Ker } \varphi)^{\perp \Delta} = \text{Im } \varphi^* \quad (\text{Im } \varphi)^{\perp \Delta} = \text{Ker } \varphi^* \quad \text{Ker } \varphi = (\text{Im } \varphi^*)^{\perp \Delta} \quad \text{Im } \varphi = (\text{Ker } \varphi^*)^{\perp \Delta} \quad (3.29)$$

The pre-abelian property of **Ste** implies

Theorem 3.15. *Each morphism $\varphi : X \rightarrow Y$ in **Ste** has basic decomposition (2.12). The operation $\varphi \mapsto \varphi^*$ of taking dual map establishes the following identities:*

$$(\text{im } \varphi)^* = \text{coim } \varphi^* \quad (\text{coim } \varphi)^* = \text{im } \varphi^* \quad (3.30)$$

$$(\text{Im } \varphi)^* = \text{Coim } \varphi^* \quad (\text{Coim } \varphi)^* = \text{Im } \varphi^* \quad (3.31)$$

Formulas (3.27) imply

Theorem 3.16. *For any morphism of stereotype spaces $\varphi : X \rightarrow Y$*

- *its kernel $\text{Ker } \varphi$ and image $\text{Im } \varphi$ are closed immediate subspaces (in X and Y respectively),*
- *its coimage $\text{Coim } \varphi$ and cokernel $\text{Coker } \varphi$ are open immediate quotient spaces (of X and Y respectively).*

Example 3.10. There exists a morphism of stereotype spaces φ such that the reduced morphism $\text{red } \varphi$ is not a bimorphism.

Proof. Let E be a space from Example 3.1, i.e. a complete locally convex space with a discontinuous linear functional $f : E \rightarrow \mathbb{C}$ which is continuous in the topology of pseudosaturation E^{Δ} of space E . The kernel $F = \text{Ker } f$ of this functional is a closed subspace in the pseudosaturation E^{Δ} of the space E , different from E^{Δ} , but in the space E the subspace F is dense. Since E is complete, we can embed it as a closed subspace into a direct product of Banach spaces, let us denote it by Y . Let $\varphi : F^{\Delta} \rightarrow Y$ be the composition of the injections

$$F^{\Delta} \subset F \subset E^{\Delta} \subset E \subset Y.$$

Since F is a closed subspace in the pseudocomplete space E^{Δ} , it is pseudocomplete. Hence, its pseudosaturation F^{Δ} is a stereotype space. On the other hand, Y is a direct product of Banach spaces, therefore it is stereotype as well. Finally, since φ is an injection, its kernel is zero, hence its coimage coincides with F^{Δ} :

$$\text{Coim } \varphi = F^{\Delta},$$

On the other hand, the image of φ is the pseudosaturation of the space $\varphi(F^{\Delta}) = F$ in Y , i.e. pseudosaturation of the space E :

$$\text{Im } \varphi = \left(\overline{\varphi(F^{\Delta})}^Y \right)^{\Delta} = E^{\Delta}.$$

Thus, the reduced morphism $\text{red } \varphi$ is just the enclosure

$$F^\Delta \subset E^\Delta,$$

and this cannot be a bimorphism, since F^Δ is closed in E^Δ , but not equal to E^Δ . Diagram (2.12) for φ takes the following form:

$$\begin{array}{ccc} F^\Delta & \xrightarrow{\varphi} & Y \\ \text{coim } \varphi \downarrow & & \uparrow \text{im } \varphi \\ F^\Delta & \xrightarrow{\text{red } \varphi} & E^\Delta \end{array}.$$

□

Nodal decomposition in \mathbf{Ste} . In [2, Theorem 4.21] it was noticed that the category \mathbf{Ste} is complete. On the other hand, from Corollaries 3.3 and 3.5 it follows that \mathbf{Ste} is well-powered and co-well-powered. Together with the existence of basic decomposition, this by Theorem 2.6 means that \mathbf{Ste} is a category with nodal decomposition:

Theorem 3.17. *In the category \mathbf{Ste} of stereotype spaces each morphism $\varphi : X \rightarrow Y$ has nodal decomposition (2.5). The operation $\varphi \mapsto \varphi^*$ of taking dual map establishes the following identities:*

$$(\text{im}_\infty \varphi)^* = \text{coim}_\infty \varphi^* \qquad (\text{coim}_\infty \varphi)^* = \text{im}_\infty \varphi^* \qquad (3.32)$$

$$(\text{Im}_\infty \varphi)^* = \text{Coim}_\infty \varphi^* \qquad (\text{Coim}_\infty \varphi)^* = \text{Im}_\infty \varphi^* \qquad (3.33)$$

As we noticed above, the basic and the nodal decomposition are connected with each other through diagram (2.13):

$$\begin{array}{ccccc} X & \xrightarrow{\varphi} & Y \\ \text{coim } \varphi \downarrow & \searrow \text{coim}_\infty \varphi & & \nearrow \text{im}_\infty \varphi & \uparrow \text{im } \varphi \\ & \text{Coim}_\infty \varphi & \xrightarrow{\text{red}_\infty \varphi} & \text{Im}_\infty \varphi & \\ & \swarrow \sigma & & \searrow \tau & \\ \text{Coim } \varphi & \xrightarrow{\text{red } \varphi} & \text{Im } \varphi \end{array}$$

where morphisms σ and τ are uniquely defined (by φ).

Example 3.11. For a morphism described in Example 3.10 diagram (2.13) has the form

$$\begin{array}{ccccc} X^\Delta & \xrightarrow{\varphi} & Y \\ \text{coim } \varphi \downarrow & \searrow \text{coim}_\infty \varphi & & \nearrow \text{im}_\infty \varphi & \uparrow \text{im } \varphi \\ & X^\Delta & \xrightarrow{\text{red}_\infty \varphi} & X^\Delta & \\ & \swarrow 1_{X^\Delta} & & \searrow \tau & \\ X^\Delta & \xrightarrow{\text{red } \varphi} & E^\Delta \end{array}$$

This shows that τ is not necessary an isomorphism. If we consider the dual map φ^* , we can conclude that σ is not necessarily an isomorphism as well.

Theorem 3.18. *For any morphism of stereotype spaces $\varphi : X \rightarrow Y$*

- *its nodal image $\text{Im}_\infty \varphi$ coincides with the envelope in Y of its set of values $\varphi(X)$:*

$$\text{Im}_\infty \varphi = \text{Env}^Y \varphi(X) \qquad (3.34)$$

- *its nodal coimage $\text{Coim}_\infty \varphi$ coincides with the imprint on X of a set of functionals $\varphi^*(Y^*)$:*

$$\text{Coim}_\infty \varphi = \text{Imp}^X \varphi^*(Y^*) \qquad (3.35)$$

Proof. By Remark 2.2, the nodal image $\mathbf{Im}_\infty \varphi$ is a projective limit of a sequence of “usual” images $\mathbf{Im} \varphi^i$ of transfinite system of morphisms, defined by the inductive rule $\varphi^{i+1} = \mathbf{red} \varphi^i$. And each space $\mathbf{Im} \varphi^i$ exactly coincides with the space E_i from the definition of the envelope $\mathbf{Env}^Y M$ in Y of the set $M = \varphi(X)$.

Similarly, the nodal coimage $\mathbf{Coim}_\infty \varphi$ is an injective limit of transfinite system of “usual” coimages $\mathbf{Coim} \varphi^i$, and each such space coincide with the space E_i from the definition of the imprint $\mathbf{Imp}^X F$ on X of the set of functionals $F = \varphi^*(Y^*)$. \square

Factorizations in **Ste.** Recall that by definition on page 49, a *factorization* of a morphism $X \xrightarrow{\varphi} Y$ is its representation as a composition $\varphi = \mu \circ \varepsilon$ of an epimorphism ε and a monomorphism μ . Theorem 2.6 implies

Theorem 3.19. *In the category **Ste** of stereotype spaces*

- (i) *each morphism φ has a factorization,*
- (ii) *among all factorizations of φ there is a minimal one $(\varepsilon_{\min}, \mu_{\min})$ and a maximal one $(\varepsilon_{\max}, \mu_{\max})$, i.e. each factorization (ε, μ) lies between them:*

$$(\varepsilon_{\min}, \mu_{\min}) \leq (\varepsilon, \mu) \leq (\varepsilon_{\max}, \mu_{\max})$$

Characterization of strong morphisms in **Ste.**

Theorem 3.20. *In the category **Ste** for a morphism $\mu : Z \rightarrow X$ the following conditions are equivalent:*

- (i) *μ is an immediate monomorphism,*
- (i)' *in diagram (3.5) the space Y is an immediate subspace in X ,*
- (ii) *μ is a strong monomorphism,*
- (ii)' *in diagram (3.5) the morphism σ is a strong monomorphism,*
- (iii) *$\mu \cong \mathbf{im}_\infty \mu$,*
- (iv) *$\mathbf{coim}_\infty \mu$ and $\mathbf{red}_\infty \mu$ are isomorphisms.*

Proof. The equivalences $(i) \iff (ii) \iff (iii) \iff (iv)$ follow from Theorem 2.3, since **Ste** is a category with nodal decomposition. In addition, Proposition 3.4 imply equivalences $(i) \iff (i)'$ and $(ii) \iff (ii)'$. \square

The dual proposition is proved by analogy:

Theorem 3.21. *In the category **Ste** for a morphism $\varepsilon : Z \rightarrow X$ the following conditions are equivalent:*

- (i) *ε is an immediate epimorphism,*
- (i)' *in diagram (3.16) the space Y is an immediate quotient space for X ,*
- (ii) *ε is a strong epimorphism,*
- (ii)' *in diagram (3.16) the morphism π is a strong epimorphism,*
- (iii) *$\varepsilon \cong \mathbf{coim}_\infty \varepsilon$,*
- (iv) *$\mathbf{im}_\infty \mu$ and $\mathbf{red}_\infty \mu$ are isomorphisms.*

Envelope and imprint in **Ste.** Since the category **Ste** is complete, well-powered, co-well-powered and has nodal decomposition, this implies the existence of some envelopes and imprints in **Ste**.

Theorem 3.22. *In the category **Ste** of stereotype spaces*

- (a) *each space X has envelopes in the classes **Epi** of all epimorphisms and **SEpi** of all strong epimorphisms with respect to arbitrary class of morphisms Φ , among which there is at least one going from X ; in addition,*
- (i) *if Φ differs morphisms on the outside in **Ste**, then the envelope in **Epi** is also an envelope in the class **Bim** of all bimorphisms:*

$$\mathbf{env}_\Phi^{\mathbf{Epi}} X = \mathbf{env}_\Phi^{\mathbf{Bim}} X,$$

- (ii) if Φ differs morphisms on the outside and is an ideal in **Ste**, then the envelope in **Epi** is also an envelope in the class **Bim** of all bimorphisms, and in any other class Ω which contains **Bim** (for example, in the class **Mor** of all morphisms):

$$\text{env}_{\Phi}^{\text{Epi}} X = \text{env}_{\Phi}^{\text{Bim}} X = \text{env}_{\Phi}^{\Omega} X = \text{env}_{\Phi} X, \quad \Omega \supseteq \text{Bim}.$$

- (b) in each space X there exist imprints of the classes **Mono** of all monomorphisms and **SMono** of all strong monomorphisms by means of arbitrary class of morphisms Φ , among which there is at least one coming to X ; in addition,

- (i) if Φ differs morphisms on the inside in **Ste**, then the imprint of **Mono** is also an imprint of the class **Bim** of all bimorphisms:

$$\text{imp}_{\Phi}^{\text{Mono}} X = \text{imp}_{\Phi}^{\text{Bim}} X.$$

- (ii) if Φ differs morphisms on the inside and is a left ideal in **Ste**, then the imprint of **Mono** is an imprint of the class **Bim** of all bimorphisms, and of any other class Ω which contains **Bim** (for example, in the class **Mor** of all morphisms):

$$\text{imp}_{\Phi}^{\text{Mono}} X = \text{imp}_{\Phi}^{\text{Bim}} X = \text{imp}_{\Phi}^{\Omega} X = \text{imp}_{\Phi} X, \quad \Omega \supseteq \text{Bim}.$$

Proof. Due to duality it is sufficient to prove (a). Let X be a stereotype space, and Φ a class of morphisms, which contains at least one going from X . Then the envelope $\text{env}_{\Phi}^{\text{Epi}} X$ exists by Theorem 2.11, and the envelope $\text{env}_{\Phi}^{\text{SEpi}} X$ by Theorem 2.21. Suppose now that Φ differs morphisms on the outside in **Ste**. Then by Theorem 1.2 the existence of envelope $\text{env}_{\Phi}^{\text{Epi}} X$ automatically implies the existence of envelope $\text{env}_{\Phi}^{\text{Bim}} X$ and their equality: $\text{env}_{\Phi}^{\text{Epi}} X = \text{env}_{\Phi}^{\text{Bim}} X$. Finally, suppose that Φ differs morphisms on the outside in **Ste** and in addition is a right ideal. Then by Theorem 1.3 the existence of envelope $\text{env}_{\Phi}^{\text{Bim}} X$ (which is already proved) implies that for any class $\Omega \supseteq \text{Bim}$ the envelope $\text{env}_{\Phi}^{\Omega} X$ also exists, and these envelopes coincide: $\text{env}_{\Phi}^{\text{Bim}} X = \text{env}_{\Phi}^{\Omega} X$. \square

Theorem 3.23. The envelope $\text{Env}^X M$ of a set M in a stereotype space X coincides with the envelope of the space¹⁰ \mathbb{C}_M in the class **Epi** of all epimorphisms of the category **Ste** with respect to the morphism $\varphi : \mathbb{C}_M \rightarrow X$, $\varphi(\alpha) = \sum_{x \in M} \alpha_x \cdot x$,

$$\text{Env}^X M = \text{Env}_{\varphi}^{\text{Epi}} \mathbb{C}_M.$$

Proof. This follows from Theorems 2.8 and 3.18:

$$\text{Env}_{\varphi}^{\text{Epi}} \mathbb{C}_M = (2.20) = \text{Im}_{\infty} \varphi = (3.34) = \text{Env}^X \varphi(\mathbb{C}_M) = \text{Env}^X \text{Span } M = \text{Env}^X M.$$

\square

Theorem 3.24. The imprint $\text{Imp}^X F$ of a set F of functionals on a stereotype space X coincides with the imprint of the class **Mono** of all monomorphisms of the category **Ste** in the space¹¹ \mathbb{C}^F by means of the morphism $\varphi : X \rightarrow \mathbb{C}^F$, $\varphi(x)^f = f(x)$, $f \in F$

$$\text{Imp}^X F = \text{Imp}_{\varphi}^{\text{Mono}} \mathbb{C}^F.$$

Proof. This follows from Theorems 2.8 and 3.18:

$$\text{Imp}_{\varphi}^{\text{Mono}} \mathbb{C}^F = (2.21) = \text{Coim}_{\infty} \varphi = (3.35) = \text{Imp}^X \varphi^*(Y^*) = \text{Imp}^X \text{Span } F = \text{Imp}^X F.$$

\square

(e) On homologies in **Ste**

As is known, in the homology theory, in opposition to the well-established methods of Abelian categories, there have always been attempts to find alternative approaches, where it is considered desirable to get rid of the Abelian property and even of the additivity with the aim to cover the widest spectrum of situations (one can make an impression of this by the works [29], [25], [40], [15], [13], [12], [35], [20], [30], [6], [7], [18], [33], [17]). We hope that the following effect will be interesting in this connection: in the (non-Abelian, but pre-Abelian) category **Ste** of stereotype spaces the standard definition of homology breaks up into two non-equivalent notions. Let us start with the following definition (taken from [17]):

¹⁰We use here the notations of [3, p.478].

¹¹The notations of [3, p.477] are used here.

- Suppose in a pre-Abelian category **K** we have a pair of morphisms $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$ which form a complex:

$$\psi \circ \varphi = 0.$$

By the definitions of kernel and cokernel, this equality defines two natural morphisms $X \xrightarrow{\varphi^{\text{Ker } \psi}} \text{Ker } \psi$ and $\text{Coker } \varphi \xrightarrow{\psi_{\text{Coker } \varphi}} Z$ such that the following diagram is commutative

$$\begin{array}{ccccc} X & \xrightarrow{\varphi} & Y & \xrightarrow{\psi} & Z \\ \downarrow \varphi^{\text{Ker } \psi} & & \searrow \text{ker } \psi & & \uparrow \psi_{\text{Coker } \varphi} \\ \text{Ker } \psi & & & & \text{Coker } \varphi \end{array}$$

The cokernel of the morphism $\varphi^{\text{Ker } \psi}$ is called the *left homology* of the pair (φ, ψ) and is denoted by

$$H_{-}(\psi : \varphi) = \text{Coker}(\varphi^{\text{Ker } \psi}). \quad (3.36)$$

and the kernel of the morphism $\psi_{\text{Coker } \varphi}$ is called the *right homology* of the pair (φ, ψ) and is denoted by

$$H_{+}(\psi : \varphi) = \text{Ker}(\psi_{\text{Coker } \varphi}). \quad (3.37)$$

The following observation belongs to folklore:

Proposition 3.11. *In a pre-Abelian category **K** for any pair of morphisms $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$ forming a complex, $\psi \circ \varphi = 0$, there exists a unique morphism $h(\psi : \varphi) : H_{-}(\psi : \varphi) \rightarrow H_{+}(\psi : \varphi)$ such that the following diagram is commutative:*

$$\begin{array}{ccccc} X & \xrightarrow{\varphi} & Y & \xrightarrow{\psi} & Z \\ \downarrow \varphi^{\text{Ker } \psi} & & \searrow \text{ker } \psi & & \uparrow \psi_{\text{Coker } \varphi} \\ \text{Ker } \psi & & & & \text{Coker } \varphi \\ \downarrow \text{coker}(\varphi^{\text{Ker } \psi}) & & & & \uparrow \text{ker}(\psi_{\text{Coker } \varphi}) \\ H_{-}(\psi : \varphi) & \xlongequal{\quad} & \text{Coker}(\varphi^{\text{Ker } \psi}) & \xrightarrow{\quad h(\psi : \varphi) \quad} & \text{Ker}(\psi_{\text{Coker } \varphi}) & \xlongequal{\quad} & H_{+}(\psi : \varphi) \end{array} \quad (3.38)$$

In each autodual category (for instance, in **Ste**) the purely categorial duality reasoning gives the following identities:

$$H_{+}(\psi : \varphi)^{\star} \cong H_{-}(\varphi^{\star} : \psi^{\star}), \quad H_{-}(\psi : \varphi)^{\star} \cong H_{+}(\varphi^{\star} : \psi^{\star}) \quad (3.39)$$

Example 3.12. In the category **Ste** of stereotype spaces the morphism $H_{-}(\psi : \varphi) \xrightarrow{h(\psi : \varphi)} H_{+}(\psi : \varphi)$ is not always an epimorphism.

Proof. Let E be a space from Example 3.1, i.e. a complete locally convex space with a discontinuous linear functional $f : E \rightarrow \mathbb{C}$, which is continuous in the topology of pseudosaturation E^{Δ} . The kernel $F = \text{Ker } f$ of this functional is a dense subspace in E , but in the space E^{Δ} it is a closed subspace, different from E^{Δ} (since $f \neq 0$). As a corollary, the natural enclosure $\sigma : F \rightarrow E$ is dense (i.e. has a dense image in E), but its pseudosaturation $\sigma^{\Delta} : F^{\Delta} \rightarrow E^{\Delta}$ does not have this property.

Let us represent E as a closed subspace in a stereotype space Y (with the topology inherited from Y ; for example, we can consider the system of Banach quotient spaces of E and say that Y is the direct product of these spaces). Let

$$\varphi : F^{\Delta} \rightarrow E^{\Delta} \rightarrow Y$$

be the corresponding composition of monomorphisms, and

$$\psi : Y \rightarrow (Y/E^{\Delta})^{\nabla}$$

the corresponding epimorphism. Then, first,

$$\begin{aligned} \text{Ker } \psi &= E^{\Delta} \\ \Downarrow \end{aligned}$$

$$\begin{aligned} \operatorname{Im} \varphi^{\operatorname{Ker} \psi} &= \left(\overline{\varphi(F)}^{E^\Delta} \right)^\Delta = \left(\overline{F}^{E^\Delta} \right)^\Delta = F^\Delta \\ &\Downarrow \\ \operatorname{Coker}(\varphi^{\operatorname{Ker} \psi}) &= (E^\Delta / F^\Delta)^\nabla \cong \mathbb{C}^\nabla = \mathbb{C}. \end{aligned}$$

And, second,

$$\begin{aligned} \operatorname{Im} \varphi &= \left(\overline{\varphi(F)}^Y \right)^\Delta = \left(\overline{F}^Y \right)^\Delta = E^\Delta, \\ &\Downarrow \\ \operatorname{Coker} \varphi &= (Y / E^\Delta)^\nabla \\ &\Downarrow \\ \psi_{\operatorname{Coker} \varphi} &= 1_{(Y / E^\Delta)^\nabla} \\ &\Downarrow \\ \operatorname{Ker}(\psi_{\operatorname{Coker} \varphi}) &= 0 \end{aligned}$$

As a result diagram (3.38) takes the form

$$\begin{array}{ccccc} F^\Delta & \xrightarrow{\varphi} & Y & \xrightarrow{\psi} & (Y / E^\Delta)^\nabla \\ \downarrow \varphi^{\operatorname{Ker} \psi} & \nearrow \ker \psi & \searrow \operatorname{coker} \varphi & \uparrow \psi_{\operatorname{Coker} \varphi} & \uparrow \ker(\psi_{\operatorname{Coker} \varphi}) \\ E^\Delta & & & & (Y / E^\Delta)^\nabla \\ \downarrow \operatorname{coker}(\varphi^{\operatorname{Ker} \psi}) & & & & \\ H_-(\psi : \varphi) & \xlongequal{\quad} & (E^\Delta / F^\Delta)^\nabla \cong \mathbb{C} & \xrightarrow{\quad h(\psi : \varphi) \quad} & 0 \xlongequal{\quad} H_+(\psi : \varphi) \end{array}$$

and clearly, $h(\psi : \varphi)$ cannot be an isomorphism. \square

§ 4 The category of stereotype algebras \mathbf{Ste}^\otimes

A stereotype space A over \mathbb{C} is called a *stereotype algebra*, if A is endowed with a structure of associative algebra over \mathbb{C} with the identity, and the multiplication is a continuous bilinear form in the following sense: for any compact set K in A and for any neighborhood of zero U in A there exists a neighborhood of zero V in A such that

$$K \cdot V \subseteq U \quad \& \quad V \cdot K \subseteq U.$$

This is equivalent to the fact that A is a monoid in the category \mathbf{Ste} of stereotype spaces with respect to one of the two natural tensor products, namely, \otimes (see details in [2]). Certainly, each stereotype algebra A is a topological algebra (but not vice versa). The class of all stereotype algebras is denoted by \mathbf{Ste}^\otimes . It is a category, where morphisms are linear, continuous, multiplicative and preserving identity maps $\varphi : A \rightarrow B$.

In contrast to the category \mathbf{Ste} of stereotype spaces, the category \mathbf{Ste}^\otimes of stereotype algebras is not additive. In addition, in \mathbf{Ste}^\otimes there arise an asymmetry between monomorphisms and epimorphisms, since epimorphisms are not inherited from \mathbf{Ste} :

Example 4.1. A morphism $\varphi : A \rightarrow B$ of stereotype algebras is a monomorphism, iff φ is an injective map (i.e. a monomorphism of stereotype spaces).

Example 4.2. On the other hand, an epimorphism $\varphi : A \rightarrow B$ of stereotype algebras not necessarily have dense image in B (i.e., not necessarily is an epimorphism of stereotype spaces). A counterexample is the enclosure of the algebra $\mathcal{R}(\mathbb{C})$ of polynomials on \mathbb{C} into the algebra $\mathcal{R}(\mathbb{C}^\times)$ of Laurent polynomials on $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ (both algebras are endowed with the strongest locally convex topology).

The following lemma will be useful in the further considerations:

Lemma 4.1. Let A and B be topological algebras (with the separately continuous multiplication), and $\varphi : A \rightarrow B$ – a linear continuous map, which is multiplicative on some dense subalgebra A_0 in A :

$$\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y), \quad x, y \in A_0.$$

Then φ is multiplicative on A :

$$\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y), \quad x, y \in A.$$

Proof. For any $x, y \in A$ we find nets $x_i, y_j \in A_0$ such that

$$x_i \xrightarrow{i \rightarrow \infty} x, \quad y_j \xrightarrow{j \rightarrow \infty} y$$

and then we have:

$$\varphi(x \cdot y) \xleftarrow{\infty \leftarrow j} \varphi(x \cdot y_j) \xleftarrow{\infty \leftarrow i} \varphi(x_i \cdot y_j) = \varphi(x_i) \cdot \varphi(y_j) \xrightarrow{i \rightarrow \infty} \varphi(x_i) \cdot \varphi(y) \xrightarrow{j \rightarrow \infty} \varphi(x) \cdot \varphi(y).$$

□

(a) Subalgebras, quotient algebras, limits and completeness of Ste^{\otimes}

Subalgebras, products and projective limits.

- Suppose B is a subset in a stereotype algebra A endowed with a structure of stereotype algebra in such a way that the set-theoretic enclosure $B \subseteq A$ is a morphism of stereotype algebras (i.e. a linear, multiplicative and preserving identity continuous map). Then the stereotype algebra B is called a *subalgebra* of the stereotype algebra A , and the set-theoretic enclosure $\sigma : B \subseteq A$ its *representing monomorphism*.
- We say that a subalgebra B of a stereotype algebra A is *closed*, if its representing monomorphism $\sigma : B \rightarrow A$ is a closed map in the sense of definition on page 74.

The following fact was stated in [2] (Theorem 10.13):

Theorem 4.1. *Let A be a stereotype algebra and B its subalgebra (in the purely algebraic sense), and at the same time a closed subspace of the locally convex space A . Then the pseudocompletion B^{Δ} is a (stereotype algebra and a) closed subalgebra in A .*

Theorem 4.2. *Each family $\{A_i; i \in I\}$ of stereotype algebras has a direct product in the category Ste^{\otimes} of stereotype algebras, and as a stereotype space this product is exactly the direct product of the family of stereotype spaces $\{A_i; i \in I\}$:*

$$\text{Ste}^{\otimes} - \prod_{i \in I} A_i = \text{Ste} - \prod_{i \in I} A_i$$

Proof. We have to verify that the direct product is the usual direct product of locally convex spaces $A = \prod_{i \in I} A_i$ with the coordinate-wise multiplication:

$$(x \cdot y)_i = x_i \cdot y_i, \quad i \in I.$$

By [2, Theorem 4.20], this is a stereotype space, so we only need to prove that the multiplication is continuous. Let U be a neighborhood of zero and K a compact set in A . We must find a neighborhood of zero V in A such that

$$V \cdot K \subseteq U, \quad K \cdot V \subseteq U.$$

It is sufficient to consider a base neighborhood of zero U , i.e.

$$U = \{x \in A : \forall i \in J \quad x_i \in U_i\}$$

where $J \subseteq I$ is a finite subset in I , and for any $i \in J$ the set U_i is a neighborhood of zero in A_i , and x_i is the projection of $x \in A$ onto A_i . If U has this form, then for any $i \in J$ we can consider the neighborhood of zero U_i in A_i , and (since A_i is a stereotype algebra) we can choose a neighborhood of zero V_i such that

$$V_i \cdot K_i \subseteq U_i, \quad K_i \cdot V_i \subseteq U_i$$

(where K_i is the projection of the compact set $K \subseteq A$ onto A_i). Then we put

$$V = \{x \in A : \forall i \in J \quad x_i \in V_i\}$$

and for each $x \in V$ and $y \in K$ we get:

$$\left(\forall i \in J \quad (x \cdot y)_i = x_i \cdot y_i \in V_i \cdot K_i \subseteq U_i \right) \implies x \cdot y \in U,$$

This means that $V \cdot K \subseteq U$. Similarly,

$$\left(\forall i \in J \quad (y \cdot x)_i = y_i \cdot x_i \in K_i \cdot V_i \subseteq U_i \right) \implies y \cdot x \in U,$$

and this means that $K \cdot V \subseteq U$.

□

Theorem 4.3. *Each covariant system $\{A_i; \pi_i^j\}$ of stereotype algebras has a projective limit in the category \mathbf{Ste}^\otimes of stereotype algebras, and as a stereotype space this limit is exactly the projective limit of the covariant system of stereotype spaces $\{A_i; \pi_i^j\}$:*

$$\mathbf{Ste}^\otimes\text{-}\varprojlim A_i = \mathbf{Ste}\text{-}\varprojlim A_i$$

Proof. By Theorem 4.2, the direct product $A = \prod_{i \in I} A_i$ with the coordinate-wise multiplication is a direct product of the family of algebras $\{A_i\}$ in \mathbf{Ste}^\otimes , and by Theorem 4.1, the subalgebra B in A , consisting of families $\{x_i; i \in I\}$ with the properties

$$x_i = \pi_i^j(x_j), \quad i \leq j \in I,$$

and endowed with the topology of pseudosaturation of the topology inherited from A , is a stereotype algebra. The same mode as in the case of stereotype spaces, prove that B is the projective limit in \mathbf{Ste}^\otimes . \square

Quotient algebras, coproducts and injective limits.

- Let A be a stereotype algebra, and let
 - 1) I be a two-sided ideal in A (as in algebra), and at the same time a closed set in A (as in a topological space), we will further call such ideals *closed ideals* in A ,
 - 2) τ be a locally convex topology on the quotient algebra A/I , such that τ is majorated by the usual quotient topology,
 - 3) B be a subspace in the completion $(A/I)^\nabla$ of the locally convex space A/I with respect to τ , such that B contains A/I and is a stereotype algebra with respect to the algebraic operations and the topology inherited from $(A/I)^\nabla$.

Then we call the stereotype algebra B the *quotient algebra of the stereotype algebra A* , and the composition $v = \sigma \circ \pi$ of the quotient map $\pi : A \rightarrow A/I$ and the natural embedding $\sigma : A/I \rightarrow B$ is called the *representing epimorphism* of the quotient algebra B .

- A quotient algebra B of a stereotype algebra A is said to be *open*, if its representing epimorphism $v : B \leftarrow A$ is an open map in the sense of definition on page 74.

The symmetry between projective and injective constructions which was obvious for stereotype space (see [2]), is preserved in some sense for stereotype algebras, but the difference is that the injective constructions in \mathbf{Ste}^\otimes become more complicated and as a corollary, the proofs become more difficult (however, the situation here is the same as for algebras in purely algebraic sense). For example, the analog of Theorem 4.1 uses the theory of modules over algebras (see proof of Theorem 10.14 in [2]):

Theorem 4.4. *Let A be a stereotype algebra and I a closed ideal in A . Then the pseudosaturation $(A/I)^\nabla$ is a stereotype algebra (and is called an *open quotient algebra of A by the ideal I*).*

Remark 4.1. In Theorem 4.4 the unitality requirement (i.e. the existence of identity) for the algebra A is unessential.

- Suppose $\{A_i; i \in I\}$ is a family of stereotype algebras. Let us construct an algebra $\coprod_{i \in I} A_i$ in the following way. First let us say that a sequence of indices $i = \{i_1, \dots, i_n\} \in I$ *alternates*, if any two neighboring elements there do not coincide:

$$\forall k = 1, \dots, n-1 \quad i_k \neq i_{k+1}.$$

The set of all alternating sequences in I of a given length $n \in \mathbb{N}$ will be denoted by I_n . Consider a stereotype space

$$A_* = \bigoplus_{n \in \mathbb{N}} \bigoplus_{i \in I_n} A_{i_1} \otimes A_{i_2} \otimes \dots \otimes A_{i_n}$$

(where \otimes is the projective tensor product described in [2]) and note that the formula

$$\left(a_{i_1} \otimes a_{i_2} \otimes \dots \otimes a_{i_m} \right) \cdot \left(b_{j_1} \otimes b_{j_2} \otimes \dots \otimes b_{j_n} \right) = \begin{cases} a_{i_1} \otimes a_{i_2} \otimes \dots \otimes a_{i_m} \otimes b_{j_1} \otimes b_{j_2} \otimes \dots \otimes b_{j_n}, & i_m \neq j_1 \\ a_{i_1} \otimes a_{i_2} \otimes \dots \otimes (a_{i_m} \cdot b_{j_1}) \otimes b_{j_2} \otimes \dots \otimes b_{j_n}, & i_m = j_1. \end{cases}$$

defines a multiplication on A_* which is continuous as a bilinear form on a stereotype space. We can factor out the algebra A_* (without identity) by the closed ideal M (here we use Remark 4.1 to Theorem 4.4) generated by elements of the form

$$1_{A_i} - 1_{A_j}, \quad i, j \in I,$$

The arising open quotient algebra $(A_*/M)^{\nabla}$ is a stereotype algebra with identity

$$1_{(A_*/M)^{\nabla}} = \pi(1_{A_i})$$

(here at the right side we mean the image of the identity 1_{A_i} in the arbitrary algebra A_i under the quotient map $\pi : A_* \rightarrow (A_*/M)^{\nabla}$). Following [24] we call the algebra $(A_*/M)^{\nabla}$ the *free product* of the algebras $\{A_i; i \in I\}$ and we denote it by $\mathbf{Ste}^{\otimes}\text{-}\coprod_{i \in I} A_i$, or by

$$\mathbf{Ste}^{\otimes}\text{-}\coprod_{i \in I} A_i = (A_*/M)^{\nabla}.$$

This is justified by the following theorem.

Theorem 4.5. *For each family $\{A_i; i \in I\}$ of stereotype algebras its free product $\mathbf{Ste}^{\otimes}\text{-}\coprod_{i \in I} A_i$ is a co-product in the category of stereotype algebras \mathbf{Ste}^{\otimes} .*

Theorem 4.6. *Each covariant system $\{A_i; \iota_i^j\}$ of stereotype algebras has an injective limit in the category \mathbf{Ste}^{\otimes} .*

Proof. This is the open quotient algebra $\left(\left(\coprod_{i \in I} A_i\right)/N\right)^{\nabla}$ of the free product $\coprod_{i \in I} A_i$ by the closed ideal N generated by elements of the form

$$\iota_i(x) - \iota_j(\iota_i^j(x)), \quad x \in A_i,$$

where $\iota_k : A_k \rightarrow \coprod_{i \in I} A_i$ are natural embeddings. □

As one can note (this is an illustration to the difference between the projective and the injective constructions in \mathbf{Ste}^{\otimes}) the injective limits in \mathbf{Ste}^{\otimes} do not necessarily coincide as stereotype spaces with the injective limits in \mathbf{Ste} . For instance for co-products we have inequality:

$$\mathbf{Ste}^{\otimes}\text{-}\coprod_{i \in I} A_i \neq \mathbf{Ste}\text{-}\coprod_{i \in I} A_i$$

(although there is a natural map from the right side to the left side). This asymmetry however disappears in the case, when the index set I is directed:

Theorem 4.7. *If $\{A_i; \iota_i^j\}$ is a covariant system of stereotype algebras over a directed set I , then the natural map*

$$\mathbf{Ste}\text{-}\varinjlim A_i \rightarrow \mathbf{Ste}^{\otimes}\text{-}\varinjlim A_i$$

between its injective limit in the category \mathbf{Ste} and the injective limit in the category \mathbf{Ste}^{\otimes} is an isomorphism of stereotype spaces:

$$\mathbf{Ste}\text{-}\varinjlim A_i \cong \mathbf{Ste}^{\otimes}\text{-}\varinjlim A_i.$$

Proof. Denote by A the injective limit of the system $\{A_i; \iota_i^j\}$ in \mathbf{Ste} :

$$A = \mathbf{Ste}\text{-}\varinjlim A_i$$

and let $\rho_i : A_i \rightarrow A$ be the corresponding morphisms of stereotype spaces:

$$\begin{array}{ccc} & A & \\ \rho_i \nearrow & & \nwarrow \rho_j \\ A_i & \xrightarrow{\iota_i^j} & A_j \end{array} \quad (4.1)$$

We will show that A has a natural structure of stereotype algebra, and with this structure A is an injective limit of the covariant system of stereotype algebras $\{A_i; \iota_i^j\}$.

1. Take $i \in I$ and note that for any $j \geq i$ the homomorphism $\iota_i^j : A_i \rightarrow A_j$ induces on A_j a structure of left A_i -module by formula

$$a \cdot_i b = \iota_i^j(a) \cdot_{A_j} b, \quad a \in A_i, \quad b \in A_j. \quad (4.2)$$

(here \cdot_i means the left multiplication by elements of A_i , and \cdot_{A_j} the multiplication in A_j). Besides this, for $i \leq j \leq k$ the maps $\iota_j^k : A_j \rightarrow A_k$ turn out to be morphisms of left A_i -modules:

$$\iota_j^k(a \cdot_i b) = (4.2) = \iota_j^k(\iota_i^j(a) \cdot_{A_j} b) = \iota_j^k(\iota_i^j(a)) \cdot_{A_k} \iota_j^k(b) = \iota_i^k(a) \cdot_{A_k} \iota_j^k(b) = (4.2) = a \cdot_i \iota_j^k(b), \quad a \in A_i, \quad b \in A_j.$$

This means that $\{A_j; j \geq i\}$ can be considered as a covariant system of left stereotype A_i -modules. By [2, Theorem 11.17], it has an injective limit, which as a stereotype space coincide with the injective limit of the system of stereotype spaces $\{A_i; j \geq i\}$. And the latter one coincides with the injective limit of all the system of stereotype spaces $\{A_i; \iota_i^j\}$, since I is directed:

$$A_i \mathbf{Ste}\text{-}\lim_{i \leq j \rightarrow \infty} A_j = \mathbf{Ste}\text{-}\lim_{i \leq j \rightarrow \infty} A_j = \mathbf{Ste}\text{-}\lim_{j \rightarrow \infty} A_j = A.$$

An important conclusion for us is that for any $i \in I$ the space A has a structure of stereotype A_i -module, and under this structure the maps in diagram (4.1) become morphisms of A_i -modules, in particular,

$$\rho_j(a \cdot_i b) = a \cdot_i \rho_j(b), \quad i \leq j, \quad a \in A_i, \quad b \in A_j. \quad (4.3)$$

2. Note then that for $i \leq j$ the structures of left A_i -module and of left A_j -module on A are coherent with each other by the identity

$$\iota_i^j(a) \cdot_j x = a \cdot_i x, \quad a \in A_i, \quad x \in A. \quad (4.4)$$

To prove this we should first consider a special case when $x = \rho_k(b)$, $b \in A_k$, $k \geq j$. We have in this situation:

$$\begin{aligned} \iota_i^j(a) \cdot_j x &= \iota_i^j(a) \cdot_j \rho_k(b) = (4.3) = \rho_k(\iota_i^j(a) \cdot_j b) = (4.2) = \rho_k\left(\iota_j^k(\iota_i^j(a)) \cdot_{A_k} b\right) = \\ &= \rho_k(\iota_i^k(a) \cdot_{A_k} b) = (4.2) = \rho_k(a \cdot_i b) = (4.3) = a \cdot_i \rho_k(b) = a \cdot_i x. \end{aligned}$$

After that let us recall that the family of spaces A_k is dense in its injective limit A (we use here the left formula of [2, (4.15)] and the fact that I is directed). This means that for any $x \in A$ there is a net $x_k \in \rho_k(A_k)$ tending to x in A :

$$x_k \xrightarrow[k \rightarrow \infty]{A} x.$$

Since for any x_k the equality (4.4) is already proved, we obtain a relation which proves (4.4) for this x :

$$\iota_i^j(a) \cdot_j x \xleftarrow[\infty \leftarrow k]{A} \iota_i^j(a) \cdot_j x_k = a \cdot_i x_k \xrightarrow[k \rightarrow \infty]{A} a \cdot_i x$$

(the possibility to take limits follows from the continuity of the multiplication in a stereotype module).

3. From the fact that A is a left A_i -module we obtain by [2, Theorem 11.2], that the formula

$$\varphi_i(a)(x) = a \cdot_i x, \quad a \in A_i, \quad x \in A,$$

defines a homomorphism of stereotype algebras

$$\varphi_i : A_i \rightarrow \mathcal{L}(A).$$

The fact that this is a homomorphism means that we have the identity

$$\varphi_i(a \cdot b) = \varphi_i(a) \circ \varphi_i(b), \quad a, b \in A_i, \quad (4.5)$$

and equality

$$\varphi_i(1_{A_i}) = \text{id}_A. \quad (4.6)$$

formula (4.3) in this style of writing turns into the identity

$$\varphi_i(a)(\rho_i(b)) = a \cdot_i \rho_i(b) = \rho_i(a \cdot_{A_i} b), \quad a, b \in A_i, \quad (4.7)$$

and formula (4.4) into the identity

$$\varphi_j(\iota_i^j(a))(x) = \varphi_i(a)(x), \quad a \in A_i, \quad x \in A,$$

which is equivalent to the equality

$$\varphi_j \circ \iota_i^j = \varphi_i, \quad i \leq j. \quad (4.8)$$

The latter one means that the following diagram in \mathbf{Ste}^\circledast is commutative:

$$\begin{array}{ccc} & \mathcal{L}(A) & \\ \varphi_i \nearrow & & \nwarrow \varphi_j \\ A_i & \xrightarrow{\iota_i^j} & A_j \end{array}$$

One can interpret this as an injective cone of the covariant system $\{A_i; \iota_i^j\}$ in the category \mathbf{Ste} of stereotype spaces. Then we can conclude that there exists a linear continuous map φ from the injective limit $A = \varinjlim A_i$ of this system into the space $\mathcal{L}(A)$ such that for any i the following diagram is commutative

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & \mathcal{L}(A) \\ \rho_i \swarrow & & \nearrow \varphi_i \\ & A_i & \end{array} \quad (4.9)$$

Let us put

$$x \cdot y = \varphi(x)(y), \quad x, y \in A, \quad (4.10)$$

and verify that this multiplication turns A into a stereotype algebra.

4. Let us note that the bilinear form $(x, y) \mapsto x \cdot y$ is continuous. Indeed, if K is a compact set in A , then its image $\varphi(K)$ is a compact set in $\mathcal{L}(A)$. Hence, $\varphi(K)$ is a compact set in the space of operators $A : A$. By [2, Theorems 5.1 and 2.5], this means that $\varphi(K)$ is equicontinuous on A . Hence for a neighborhood of zero W in A there is a neighborhood of zero V in A such that

$$K \cdot V = \varphi(K)(V) \subseteq W.$$

On the other hand, for any compact set K and for any neighborhood of zero W in A the set $W \odot K$ is a neighborhood of zero in $\mathcal{L}(A)$, hence from the continuity of φ it follows that there is a neighborhood of zero V in A such that

$$\varphi(V) \subseteq W \odot K,$$

and this is equivalent to the inclusion

$$V \cdot K = \varphi(V)(K) \subseteq V.$$

5. Besides this, the formula

$$1_A = \rho_i(1_{A_i}) \quad (4.11)$$

defines some element of the space A , so if $i \leq j$, then

$$\rho_j(1_{A_j}) = \rho_j(\iota_i^j(1_{A_i})) = \rho_i(1_{A_i}).$$

At the same time the chain

$$\varphi(1_A) = \varphi(\rho_i(1_{A_i})) = \varphi_i(1_{A_i}) = (4.6) = \text{id}_A \quad (4.12)$$

implies that this element is the identity for the multiplication (4.10): first, for any $y \in A$ we have

$$1_A \cdot y = \varphi(1_A)(y) = \text{id}_A(y) = y.$$

And, second, for any $x \in A$ we can find a net $a_k \in A_k$ such that

$$\rho_k(a_k) \xrightarrow[k \rightarrow \infty]{A} x,$$

and by the already proven continuity of the multiplication in A , we have:

$$\begin{aligned} x \cdot 1_A &\xleftarrow[\infty \leftarrow k]{A} \rho_k(a_k) \cdot 1_A = \rho_k(a_k) \cdot \rho_k(1_{A_k}) = (4.10) = \varphi(\rho_k(a_k))(\rho_k(1_{A_k})) = \\ &= \varphi_k(a_k)(\rho_k(1_{A_k})) = (4.7) = \rho_k(a_k) \cdot_{A_k} 1_{A_k} = \rho_k(a_k) \xrightarrow[k \rightarrow \infty]{A} x, \end{aligned}$$

Thus,

$$x \cdot 1_A = x.$$

6. Now we notice that the map ρ_i in (4.9) must be a homomorphism of algebras. Indeed it turns identity into identity just by the definition of 1_A in (4.11). On the other hand, it preserves multiplication since for all $a, b \in A_i$

$$\rho_i(a \cdot_{A_i} b) = (4.7) = \varphi_i(a)(\rho_i(b)) = (4.9) = \varphi(\rho_i(a))(\rho_i(b)) = (4.10) = \rho_i(a) \cdot \rho_i(b). \quad (4.13)$$

7. The same for the map φ . The preserving of identities were already stated in the chain (4.11). And to prove multiplicativity we first have to note the formula

$$\varphi(\rho_i(a) \cdot \rho_j(b)) = \varphi(\rho_i(a)) \circ \varphi(\rho_j(b)), \quad i, j \in I, \quad a \in A_i, \quad b \in A_j \quad (4.14)$$

Indeed, for $k \in I$ such that $k \geq i$ and $k \geq j$ we have:

$$\begin{aligned} \varphi(\rho_i(a) \cdot \rho_j(b)) &= (4.1) = \varphi(\rho_k(\iota_i^k(a)) \cdot \rho_k(\iota_j^k(b))) = (4.13) = \varphi(\rho_k(\iota_i^k(a) \cdot \iota_j^k(b))) = \\ &= (4.9) = \varphi_k(\iota_i^k(a) \cdot \iota_j^k(b)) = (4.5) = \varphi_k(\iota_i^k(a)) \circ \varphi_k(\iota_j^k(b)) = (4.9) = \\ &= \varphi(\rho_k(\iota_i^k(a))) \circ \varphi(\rho_k(\iota_j^k(b))) = (4.1) = \varphi(\rho_i(a)) \circ \varphi(\rho_j(b)) \end{aligned}$$

Then we take $x, y \in A$ and find $a_i \in A_i$ and $b_j \in A_j$ such that

$$\rho_i(a_i) \xrightarrow{i \rightarrow \infty} x, \quad \rho_j(b_j) \xrightarrow{j \rightarrow \infty} y.$$

We obtain:

$$\begin{aligned} \varphi(x \cdot y) &\xrightarrow[\infty \leftarrow i]{\mathcal{L}(A)} \varphi(\rho_i(a_i) \cdot y) \xrightarrow[\infty \leftarrow j]{\mathcal{L}(A)} \varphi(\rho_i(a_i) \cdot \rho_j(b_j)) = (4.14) = \\ &= \varphi(\rho_i(a_i)) \circ \varphi(\rho_j(b_j)) \xrightarrow[j \rightarrow \infty]{\mathcal{L}(A)} \varphi(\rho_i(a_i)) \circ \varphi(y) \xrightarrow[i \rightarrow \infty]{\mathcal{L}(A)} \varphi(x) \circ \varphi(y), \end{aligned}$$

hence,

$$\varphi(x \cdot y) = \varphi(x) \circ \varphi(y).$$

This formula proves in addition the associativity of the multiplication in A ,

$$x \cdot (y \cdot z) = \varphi(x)(y \cdot z) = \varphi(x)(\varphi(y)(z)) = (\varphi(x) \circ \varphi(y))(z) = \varphi(x \cdot y)(z) = (x \cdot y) \cdot z,$$

and this was the last what we needed to understand that A is a stereotype algebra.

8. We only have to verify that the cone of algebras $\{A_i; \rho_i\}$ is an injective limit of the covariant system of algebras $\{A_i; \iota_i^j\}$. Let $\{B_i; \sigma_i\}$ be another cone of algebras. Since it is also a cone of stereotype spaces, there exists a unique linear continuous map $\sigma : A \rightarrow B$ such that the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\sigma} & B \\ \rho_i \swarrow & & \nearrow \sigma_i \\ & A_i & \end{array} \quad (4.15)$$

We must check that σ is a homomorphism of algebras. The preserving of identities follows from the fact that all σ_i preserve identity:

$$\sigma(1_A) = \sigma(\rho_i(1_{A_i})) = \sigma_i(1_{A_i}) = 1_B.$$

For proving the multiplicativity we note first the following identity:

$$\sigma(\rho_i(a) \cdot \rho_j(b)) = \sigma(\rho_i(a)) \cdot \sigma(\rho_j(b)), \quad i, j \in I, \quad a \in A_i, \quad b \in A_j, \quad (4.16)$$

This can be proved by the same reasoning as (4.14) above: take $k \in I$ such that $k \geq i$ and $k \geq j$, then

$$\begin{aligned} \sigma(\rho_i(a) \cdot \rho_j(b)) &= (4.1) = \sigma(\rho_k(\iota_i^k(a)) \cdot \rho_k(\iota_j^k(b))) = (4.13) = \sigma(\rho_k(\iota_i^k(a) \cdot \iota_j^k(b))) = \\ &= (4.15) = \sigma_k(\iota_i^k(a) \cdot \iota_j^k(b)) = \sigma_k(\iota_i^k(a)) \cdot \sigma_k(\iota_j^k(b)) = (4.15) = \\ &= \sigma(\rho_k(\iota_i^k(a))) \cdot \sigma(\rho_k(\iota_j^k(b))) = (4.1) = \sigma(\rho_i(a)) \cdot \sigma(\rho_j(b)) \end{aligned}$$

After that we take $x, y \in A$ and choose $a_i \in A_i$ and $b_j \in A_j$ such that

$$\rho_i(a_i) \xrightarrow{i \rightarrow \infty} x, \quad \rho_j(b_j) \xrightarrow{j \rightarrow \infty} y.$$

We obtain:

$$\begin{aligned} \sigma(x \cdot y) &\xrightarrow[\infty \leftarrow i]{B} \sigma(\rho_i(a_i) \cdot y) \xrightarrow[\infty \leftarrow j]{B} \sigma(\rho_i(a_i) \cdot \rho_j(b_j)) = (4.16) = \\ &= \sigma(\rho_i(a_i)) \cdot \sigma(\rho_j(b_j)) \xrightarrow[j \rightarrow \infty]{B} \sigma(\rho_i(a_i)) \cdot \sigma(y) \xrightarrow[i \rightarrow \infty]{B} \sigma(x) \cdot \sigma(y), \end{aligned}$$

and thus,

$$\sigma(x \cdot y) = \sigma(x) \cdot \sigma(y).$$

□

\mathbf{Ste}^\otimes as a determined category. Theorems 4.3 and 4.6 imply

Theorem 4.8. *The category \mathbf{Ste}^\otimes of stereotype algebras is determined (and hence, complete).*

(b) Nodal decomposition, envelope and imprint in \mathbf{Ste}^\otimes

Discerning properties of strong epimorphisms in \mathbf{Ste}^\otimes .

Theorem 4.9. *For a morphism of stereotype algebras $\varepsilon : A \rightarrow B$ the following conditions are equivalent:*

- (i) ε is an immediate epimorphism in \mathbf{Ste}^\otimes ,
- (ii) ε is a strong epimorphism in \mathbf{Ste}^\otimes ,
- (iii) ε is an immediate epimorphism in \mathbf{Ste} ,
- (iv) ε is a strong epimorphism in \mathbf{Ste} .

Proof. Let us note that the connections (i) \Leftarrow (ii) and (iii) \Leftrightarrow (iv) are already known. So it is sufficient to prove (i) \Rightarrow (iii) and (ii) \Leftarrow (iv).

1. Let us start with (i) \Rightarrow (iii). Let $\varepsilon : A \rightarrow B$ be an immediate epimorphism in \mathbf{Ste}^\otimes . Consider its minimal factorization in \mathbf{Ste} , i.e. a diagram with linear continuous maps

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon} & B \\ & \searrow \text{coim}_\infty \varepsilon & \nearrow \mu \\ & \text{Coim}_\infty \varepsilon & \end{array}$$

where $\text{Coim}_\infty \varepsilon$ is the nodal coimage in \mathbf{Ste} . Our aim is to show that $\text{Coim}_\infty \varepsilon$ has a structure of stereotype algebra, under which the morphisms $\text{coim}_\infty \varepsilon$ and μ become morphisms in \mathbf{Ste}^\otimes – this will mean that the epimorphism $\text{coim}_\infty \varepsilon$ is a mediator for ε in the category \mathbf{Ste}^\otimes , and, since ε is an immediate epimorphism, μ must be an isomorphism in \mathbf{Ste}^\otimes , and hence in \mathbf{Ste} as well. This allows to conclude that the epimorphism ε is isomorphic in \mathbf{Ste} to the epimorphism $\text{Coim}_\infty \varepsilon$, which is an immediate epimorphism in \mathbf{Ste} , and thus, ε is also an immediate epimorphism in \mathbf{Ste} .

The existence of the structure of stereotype algebra on $\text{Coim}_\infty \varepsilon$ follows from Theorems 4.4 and 4.7: on the one hand, any operation of the form $A' \mapsto (A'/I)^\nabla$ (where I is a closed two-sided ideal in A') turns each stereotype algebra A' into a stereotype algebra, and on the other hand, the injective limit in \mathbf{Ste} of the system of stereotype algebras that one can form from A in this way, is a stereotype algebra. Theorem 4.7 implies also that the natural map of A into this injective limit $\text{Coim}_\infty \varepsilon$ is a morphism of stereotype algebras.

It remains to check that μ is a morphism of stereotype algebras as well, i.e. it is multiplicative and it preserves identity. Preserving identity follows from the same property for ε and $\text{coim}_\infty \varepsilon(1_A)$:

$$\mu(1_C) = \mu(\text{coim}_\infty \varepsilon(1_A)) = \varepsilon(1_A) = 1_B.$$

The multiplicativity of μ on the subalgebra $\text{coim}_\infty \varepsilon(A)$ follows from the multiplicativity of ε and $\text{coim}_\infty \varepsilon(1_A)$: for any $a, b \in A$ we have

$$\mu(\text{coim}_\infty \varepsilon(a) \cdot \text{coim}_\infty \varepsilon(b)) = \mu(\text{coim}_\infty \varepsilon(a \cdot b)) = \varepsilon(a \cdot b) = \varepsilon(a) \cdot \varepsilon(b) = \mu(\text{coim}_\infty \varepsilon(a)) \cdot \mu(\text{coim}_\infty \varepsilon(b))$$

After that we should recall that $\text{coim}_\infty \varepsilon$ is an epimorphism in \mathbf{Ste} , so the algebra $\text{coim}_\infty \varepsilon(A)$ is dense in $\text{Coim}_\infty \varepsilon$. Hence, by Lemma 4.1, μ must be multiplicative on $\text{Coim}_\infty \varepsilon$.

2. Let us now prove (ii) \Leftarrow (iv). Suppose $\varepsilon : A \rightarrow B$ is a strong epimorphism in \mathbf{Ste} . Consider a diagram in \mathbf{Ste}^\otimes

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon} & B \\ \alpha \downarrow & & \downarrow \beta \\ C & \xrightarrow{\mu} & D \end{array}$$

where μ is a monomorphism. It can be considered as a diagram in \mathbf{Ste} , and since μ is a monomorphism in \mathbf{Ste} (by Example 4.1), and ε a strong epimorphism in \mathbf{Ste} , there must exist a morphism δ in \mathbf{Ste} (i.e. a linear continuous map) such that the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon} & B \\ \alpha \downarrow & \swarrow \delta & \downarrow \beta \\ C & \xrightarrow{\mu} & D \end{array}$$

It remains to check that the map δ is a homomorphism of algebras. Preserving identity follows from monomorphy of μ :

$$\mu(1_C) = 1_D = \beta(\varepsilon(1_A)) = \mu(\delta(\varepsilon(1_A))) = \mu(\delta(1_B)) \implies 1_C = \delta(1_B).$$

By the same reason δ is multiplicative on the subalgebra $\varepsilon(A)$: for each $a, b \in A$

$$\begin{aligned} \mu(\delta(\varepsilon(a \cdot b))) &= \beta(\varepsilon(a \cdot b)) = \beta(\varepsilon(a) \cdot \varepsilon(b)) = \mu(\delta(\varepsilon(a))) \cdot \mu(\delta(\varepsilon(b))) = \mu(\delta(\varepsilon(a)) \cdot \delta(\varepsilon(b))) \\ &\Downarrow \\ \delta(\varepsilon(a \cdot b)) &= \delta(\varepsilon(a)) \cdot \delta(\varepsilon(b)). \end{aligned}$$

After that the multiplicativity of δ on B follows from Lemma 4.1. \square

Theorem 4.10. *If a morphism of stereotype algebras $\varphi : A \rightarrow B$ is not a monomorphism, then there exists a decomposition $\varphi = \varphi' \circ \varepsilon$, where ε is a strong epimorphism, but not an isomorphism.*

Proof. If φ is not a monomorphism, then its kernel $I = \text{Ker } \varphi$ is a nonzero closed ideal in A . By Theorem 4.4 the quotient space $(A/I)^\nabla$ is a stereotype algebra. The homomorphism of algebras φ can be lifted to some homomorphism of algebras $\psi : A/I \rightarrow B$, which by definition of usual quotient topology is a continuous map:

$$\begin{array}{ccc} A & \xrightarrow{\pi} & A/I \\ & \searrow \varphi & \downarrow \psi \\ & & B \end{array}$$

Since the space B is pseudocomplete, the map ψ can be extended to a continuous map $\varphi' : (A/I)^\nabla \rightarrow B$

$$\begin{array}{ccccc} A & \xrightarrow{\pi} & A/I & \xrightarrow{\nabla_{A/I}} & (A/I)^\nabla \\ & \searrow \varphi & \downarrow \psi & \swarrow \varphi' & \\ & & B & & \end{array}$$

By Theorem 4.9 the map $v = \nabla_{A/I} \circ \pi : A \rightarrow (A/I)^\nabla$ is a strong epimorphism of stereotype algebras, so we only have to verify that φ' is a homomorphism of algebras. It preserves identity since $1_{(A/I)^\nabla} = 1_{A/I}$:

$$\varphi'(1_{(A/I)^\nabla}) = \psi(1_{A/I}) = 1_B$$

And its multiplicativity follows from Lemma 4.1, since ψ is multiplicative. \square

Discerning properties of strong monomorphisms in \mathbf{Ste}^\otimes .

Lemma 4.2. *Let A be a stereotype algebra and B a subalgebra in A (in the purely algebraic sense). Then the envelope $\text{Env}^A B$ of the set B in the stereotype space A is a stereotype algebra.*

Proof. This follows from the completeness of the category \mathbf{Ste}^\otimes (Theorem 4.8) and from the fact that the pseudosaturation of closure \overline{C}^Δ of any subalgebra C in A is always a stereotype algebra by Theorem 4.1. \square

Lemma 4.3. *In the category \mathbf{Ste}^\otimes of stereotype algebras the immediate monomorphisms coincide with the strong monomorphisms.*

Proof. We already noticed (the property 2° on page 14) that each strong monomorphism is an immediate monomorphism, so we have to verify that in \mathbf{Ste}^\otimes the inverse is also true. Let $\mu : C \rightarrow D$ be an immediate monomorphism of stereotype algebras. Consider a diagram

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon} & B \\ \alpha \downarrow & & \downarrow \beta \\ C & \xrightarrow{\mu} & D \end{array}$$

where ε is an epimorphism. Consider the subset in $\mu(C) \cup \beta(B)$ in D . Let $\text{alg}(\mu(C) \cup \beta(B))$ be the subalgebra (in the purely algebraic sense) in D generated by $\mu(C) \cup \beta(B)$, and $R = \text{Env}^D(\text{alg}(\mu(C) \cup \beta(B)))$ the envelope

of the set $\mathbf{alg}(\mu(C) \cup \beta(B))$ in D (in the sense of the definition on page 82). By Lemma 4.2, R is a stereotype algebra. Let $\sigma : R \rightarrow D$ denote its natural enclosure in D . Since $\mu(C) \subseteq R$, and R is an immediate subspace in D , the morphism of stereotype spaces μ can be factored through the morphism of stereotype spaces $\sigma : R \rightarrow D$,

$$\mu = \sigma \circ \pi$$

Here π must be multiplicative, since from the identities

$$\sigma(\pi(x \cdot y)) = \mu(x \cdot y) = \mu(x) \cdot \mu(y) = \sigma(\pi(x)) \cdot \sigma(\pi(y)) = \sigma(\pi(x) \cdot \pi(y))$$

imply by monomorphy of σ the identity

$$\pi(x \cdot y) = \pi(x) \cdot \pi(y).$$

So we can conclude that π is a morphism of stereotype algebras. Similarly, the enclosure $\beta(B) \subseteq R$ implies that the morphism of stereotype spaces β can be factored through the morphism of stereotype spaces $\sigma : R \rightarrow D$,

$$\beta = \sigma \circ \rho$$

and again the monomorphy of σ implies that ρ is a morphism of stereotype algebras.

So we obtain a diagram in the category \mathbf{Ste}^{\otimes} :

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon} & B \\ \alpha \downarrow & \nearrow \rho & \downarrow \beta \\ & R & \\ \nearrow \pi & \searrow \sigma & \\ C & \xrightarrow{\mu} & D \end{array}$$

Let us show that π is an epimorphism (in \mathbf{Ste}^{\otimes}). Let $\zeta, \eta : R \rightrightarrows T$ be two parallel morphisms of stereotype algebras. Then the equality

$$\zeta \circ \pi = \eta \circ \pi$$

implies, on the one hand, the identity

$$\zeta|_{\pi(C)} = \eta|_{\pi(C)},$$

and, on the other hand, it implies the chain

$$\zeta \circ \rho \circ \varepsilon = \zeta \circ \pi \circ \alpha = \eta \circ \pi \circ \alpha = \eta \circ \rho \circ \underset{\text{End}}{\varepsilon} \implies \zeta \circ \rho = \eta \circ \rho \implies \zeta|_{\rho(B)} = \eta|_{\rho(B)}.$$

Together they give

$$\zeta|_{\pi(C) \cup \rho(B)} = \eta|_{\pi(C) \cup \rho(B)} \implies \zeta|_{\mathbf{alg}(\pi(C) \cup \rho(B))} = \eta|_{\mathbf{alg}(\pi(C) \cup \rho(B))}.$$

Let us recall that formally R is a subset in B , so the set $\mathbf{alg}(\pi(C) \cup \rho(B))$ formally coincides with the set $\mathbf{alg}(\mu(C) \cup \beta(B))$. As a corollary, $\mathbf{alg}(\pi(C) \cup \rho(B)) = \mathbf{alg}(\mu(C) \cup \beta(B))$ is dense in R , and we obtain that $\zeta = \eta$.

This proves that π is an epimorphism of stereotype algebras. Thus, μ is decomposed into a composition of an epimorphism π and a monomorphism σ . Since μ is an immediate monomorphism, π , being a mediator, must be an isomorphism. Now we can put $\delta = \pi^{-1} \circ \rho$, and we obtain the required diagram

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon} & B \\ \alpha \downarrow & \nearrow \delta & \downarrow \beta \\ & C & \\ \nearrow \mu & & \\ C & \xrightarrow{\mu} & D \end{array}$$

□

Theorem 4.11. *If a morphism of stereotype algebras $\varphi : A \rightarrow B$ is not an epimorphism, then there exists a decomposition $\varphi = \lambda \circ \varphi'$ (in \mathbf{Ste}^{\otimes}), where λ is a strong monomorphism, but not an isomorphism.*

Proof. 1. Denote by P the envelope in B of the set $\varphi(A)$:

$$P = \text{Env}^B \varphi(A).$$

By Lemma 4.2, P is a stereotype algebra, and the set-theoretic enclosure $\iota : P \rightarrow B$ is a monomorphism of stereotype algebras (and an immediate monomorphism of stereotype spaces). Let Φ be the class of all factorizations of the morphism ι in \mathbf{Ste}^{\otimes} ,

$$\begin{array}{ccc} P & \xrightarrow{\iota} & B \\ \text{Epi} \ni \pi \searrow & & \nearrow \mu \in \text{Mono} \\ & X & \end{array} \quad (4.17)$$

where the algebra X as a set lies between P and B :

$$P \subseteq X \subseteq B \quad (4.18)$$

This class is not empty, since it contains the factorization $\iota = \iota \circ 1$, and it is full in the class of all factorizations (i.e. each factorization of ι is isomorphic to some factorization from Φ). Every factorization from Φ is uniquely defined by the set X in B and a topology on X , i.e. by a subspace X in the topological space B . Since all subspaces of a given topological spaces form a set, we obtain that Φ must be a set (not just a class). For simplicity we can conceive Φ as just a set of subalgebras X in B satisfying (4.18) and endowed a topology that turns X into stereotype algebras in such a way that the enclosures (4.18) are continuous maps (this will mean that they are morphisms of stereotype algebras). For any $X \in \Phi$ the set-theoretic enclosures $P \subseteq X$ and $X \subseteq B$ will be denoted by π_X and μ_X . Thus, diagram (4.17) turns into diagram

$$\begin{array}{ccc} P & \xrightarrow{\iota} & B \\ \pi_X \searrow & & \nearrow \mu_X \\ & X & \end{array} \quad (4.19)$$

Let Y be the union of all sets X :

$$Y = \bigcup_{X \in \Phi} X,$$

then Q the envelope of the subalgebra $\text{alg } Y$ in the stereotype space B ,

$$Q = \text{Env}^B \text{alg } Y$$

and \varkappa and λ the enclosures $P \subseteq Q$ and $Q \subseteq B$ respectively:

$$\begin{array}{ccc} P & \xrightarrow{\iota} & B \\ \varkappa \searrow & & \nearrow \lambda \\ & Q & \end{array}$$

By Lemma 4.2, Q is a stereotype algebra, and this means that \varkappa and λ are (mono)morphisms of stereotype algebras. For any $X \in \Phi$ we denote by σ_X the enclosure $X \subseteq Q$. The topology of X majorizes the topology of Q , hence σ_X is a continuous map, and we obtain a diagram in the category \mathbf{Ste}^{\otimes} :

$$\begin{array}{ccc} P & \xrightarrow{\iota} & B \\ \pi_X \searrow & & \nearrow \mu_X \\ & X & \\ \varkappa \searrow & \sigma_X \downarrow & \nearrow \lambda \\ & Q & \end{array} \quad (4.20)$$

2. Let us show now that \varkappa is (not only a monomorphism, but also) an epimorphism of stereotype algebras. Indeed, for any two morphisms $\zeta, \eta : Q \rightrightarrows T$ we have the following chain:

$$\begin{aligned} \zeta \circ \varkappa &= \eta \circ \varkappa \\ \Downarrow \end{aligned}$$

$$\begin{aligned}
\forall X \in \Phi \quad \zeta \circ \sigma_X \circ \pi_X = \eta \circ \sigma_X \circ \pi_X \underset{\text{End}}{\implies} \quad \forall X \in \Phi \quad \zeta \circ \sigma_X = \eta \circ \sigma_X \implies \quad \forall X \in \Phi \quad \zeta|_X = \eta|_X \implies \\
\implies \quad \zeta|_Y = \zeta|_{\bigcup_{X \in \Phi} X} = \eta|_{\bigcup_{X \in \Phi} X} = \eta|_Y \implies \quad \zeta|_{\text{alg } Y} = \eta|_{\text{alg } Y} \implies \quad \zeta = \zeta|_Q = \eta|_Q = \eta
\end{aligned}$$

(the last implication follows from the fact the the vector space $\text{alg } Y$ is dense in its envelope).

3. Let us show that $\lambda : Q \rightarrow B$ is an immediate monomorphism (in \mathbf{Ste}^{\otimes}). Suppose $\lambda = \lambda' \circ \varepsilon$ is its arbitrary factorization. Denote by R the range of ε (and the domain of λ'), then we have a diagram:

$$\begin{array}{ccc}
P & \xrightarrow{\iota} & B \\
\searrow \varkappa & & \nearrow \lambda \\
& Q & \\
\downarrow \varepsilon & & \downarrow \lambda' \\
& R &
\end{array}
\quad (4.21)$$

The morphism $\varepsilon \circ \varkappa$ is an epimorphism (as a composition of two epimorphisms), so the decomposition $\iota = \lambda' \circ (\varepsilon \circ \varkappa)$ is a factorization of ι . As a corollary, it is isomorphic to some standard factorization $\iota = \mu_X \circ \pi_X$ for some $X \in \Phi$:

$$\begin{array}{ccc}
P & \xrightarrow{\iota} & B \\
\searrow \varkappa & & \nearrow \lambda \\
& Q & \\
\downarrow \varepsilon & & \downarrow \lambda' \\
& R & \\
\downarrow \text{---} & & \downarrow \text{---} \\
& X &
\end{array}$$

(here the dashed arrow is some isomorphism of stereotype algebras). So from the very beginning we can think that in (4.21) some $X \in \Phi$ stands instead of R :

$$\begin{array}{ccc}
P & \xrightarrow{\iota} & B \\
\searrow \varkappa & & \nearrow \lambda \\
& Q & \\
\downarrow \varepsilon & & \downarrow \lambda' \\
& X &
\end{array}$$

Here every arrow is a set-theoretic enclosure, and the topology on the beginning of the arrow majorizes the topology on its end. In particular, the arrow ε means that Q is a subset of X , and the topology of Q majorizes the topology of X . But on the other hand the arrow σ_X in diagram (4.20) means that on the contrary X is a subset in Q , and the topology of X majorizes the topology of Q . Together this means that X and Q coincide with the topologies:

$$X \cong Q.$$

In particular, ε is an isomorphism, and this is what we had to verify.

4. Since λ is an immediate monomorphism, by Lemma 4.3, we obtain that λ is a strong monomorphism.

5. Note that since $\varphi(A) \subseteq P$, the morphism φ is factored through P :

$$\varphi = \iota \circ \theta,$$

for some morphism $\theta : A \rightarrow P$. We obtain a diagram in \mathbf{Ste}^{\otimes} :

$$\begin{array}{ccc}
A & \xrightarrow{\varphi} & B \\
\downarrow \theta & \nearrow \iota & \downarrow \lambda \\
P & \xrightarrow{\varkappa} & Q
\end{array}$$

We see now that λ cannot be an isomorphism, since otherwise φ would be an epimorphism, as a composition of two epimorphisms θ and \varkappa , and an isomorphism λ . So if we put $\varphi' = \varkappa \circ \theta$, we obtain a decomposition $\varphi = \lambda \circ \varphi'$, where λ is a strong monomorphism, but not an isomorphism. \square

Nodal decomposition in \mathbf{Ste}^\otimes . Let us notice the following two properties of the category \mathbf{Ste}^\otimes .

Theorem 4.12. *The category \mathbf{Ste}^\otimes of stereotype algebras is well-powered.*

Proof. A morphism $\mu : A \rightarrow B$ in \mathbf{Ste}^\otimes is a monomorphism in \mathbf{Ste}^\otimes iff it is a monomorphism in \mathbf{Ste} , and the latter category is co-well-powered. \square

Theorem 4.13. *The category \mathbf{Ste}^\otimes of stereotype algebras is strongly co-well-powered.*

Proof. By Theorem 4.9, a morphism $\varepsilon : A \rightarrow B$ in \mathbf{Ste}^\otimes is a strong epimorphism in \mathbf{Ste}^\otimes iff it is a strong epimorphism in \mathbf{Ste} , and the latter category is co-well-powered. \square

On the other hand, as we already know the category \mathbf{Ste}^\otimes is complete (by Theorem 4.8), and in \mathbf{Ste}^\otimes the strong epimorphisms discern monomorphisms, and the strong monomorphisms discern epimorphisms (Theorems 4.10 and 4.11). Thus, we can apply Theorem 2.4, and we get

Theorem 4.14. *In the category \mathbf{Ste}^\otimes of stereotype algebras each morphism $\varphi : X \rightarrow Y$ has a nodal decomposition (2.5).*

Remark 4.2. Theorem 4.9 implies in addition that the nodal coimage $\mathbf{Coim}_\infty \varphi$ in \mathbf{Ste}^\otimes coincides with the nodal coimage in \mathbf{Ste} , and as a corollary with the imprint (as a quotient space of a stereotype space) on X of a set of functionals $\varphi^*(Y^*)$:

$$\mathbf{Coim}_\infty \varphi = \mathbf{Imp}^X \varphi^*(Y^*) \quad (4.22)$$

For the nodal image $\mathbf{Im}_\infty \varphi$ the analogous proposition is not true.

Theorem 4.15. *For each morphism $\varphi : A \rightarrow B$ in the category \mathbf{Ste}^\otimes of stereotype algebras its nodal decomposition $\varphi = \mathbf{im}_\infty \varphi \circ \mathbf{red}_\infty \varphi \circ \mathbf{coim}_\infty \varphi$ in the category \mathbf{Ste} of stereotype spaces is a decomposition (not necessarily, nodal) in the category \mathbf{Ste}^\otimes .*

Proof. We need to see here that the stereotype spaces $\mathbf{Coim}_\infty \varphi$ and $\mathbf{Im}_\infty \varphi$ have natural structure of stereotype algebras, and that the morphisms of stereotype spaces $\mathbf{coim}_\infty \varphi : A \rightarrow \mathbf{Coim}_\infty \varphi$, $\mathbf{red}_\infty \varphi : \mathbf{Coim}_\infty \varphi \rightarrow \mathbf{Im}_\infty \varphi$, $\mathbf{im}_\infty \varphi : \mathbf{Im}_\infty \varphi \rightarrow B$, are morphisms of stereotype algebras (i.e., homomorphisms of algebras). This follows from the way of constructing $\mathbf{Coim}_\infty \varphi$ and $\mathbf{Im}_\infty \varphi$: since $\varphi : A \rightarrow B$ is a morphism of stereotype algebras, its reduced morphism $\varphi^1 = \mathbf{red} \varphi : \mathbf{Coim} \varphi \rightarrow \mathbf{Im} \varphi$ is also a morphism of stereotype algebras (together with the morphisms $\mathbf{coim} \varphi : A \rightarrow \mathbf{Coim} \varphi$ and $\mathbf{im} \varphi : \mathbf{Im} \varphi \rightarrow B$). By the same reason the second reduced morphism $\varphi^2 = \mathbf{red} \varphi^1$ must be a morphism of stereotype algebras, and so on. We have to organize a transfinite induction by the degree of this operation, and we will obtain that the nodal coimage $\mathbf{Coim}_\infty \varphi$ is a stereotype algebra (as an injective limit of stereotype algebras $\mathbf{Coim} \varphi^i$), the nodal image $\mathbf{Im}_\infty \varphi$ is a stereotype algebra (as a projective limit of stereotype algebras $\mathbf{Im} \varphi^i$), and the morphisms $\mathbf{coim}_\infty \varphi : A \rightarrow \mathbf{Coim}_\infty \varphi$, $\mathbf{red}_\infty \varphi : \mathbf{Coim}_\infty \varphi \rightarrow \mathbf{Im}_\infty \varphi$, $\mathbf{im}_\infty \varphi : \mathbf{Im}_\infty \varphi \rightarrow B$ are homomorphisms of algebras. \square

Envelopes and imprints in \mathbf{Ste}^\otimes . Since it is not clear whether the category \mathbf{Ste}^\otimes is co-well-powered, in the analog of Theorem 3.22 in the case of envelopes in \mathbf{Epi} we have to require that the class Φ of test morphisms is a set (so that in the proof one could replace Theorem 2.11 by Theorem 2.9):

Theorem 4.16. *In the category \mathbf{Ste}^\otimes of stereotype algebras*

(a) *each algebra A has envelope in the class \mathbf{Epi} of all epimorphisms (respectively, in the class \mathbf{SEpi} of all strong epimorphisms) of the category \mathbf{Ste}^\otimes with respect to an arbitrary set (respectively, class) of morphisms Φ , among which there is at least one going from A ; in addition,*

(i) *if Φ differs morphisms on the outside in \mathbf{Ste}^\otimes , then the envelope in the class \mathbf{Epi} is an envelope in the class \mathbf{Bim} of all bimorphisms:*

$$\mathbf{env}_\Phi^{\mathbf{Epi}} A = \mathbf{env}_\Phi^{\mathbf{Bim}} A,$$

(ii) *if Φ differs morphisms on the outside and is a right ideal in \mathbf{Ste}^\otimes , then the envelope in the class \mathbf{Epi} is an envelope in the class \mathbf{Bim} of all bimorphisms, and in any other class Ω which contains \mathbf{Bim} (for example, in the class \mathbf{Mor} of all morphisms):*

$$\mathbf{env}_\Phi^{\mathbf{Epi}} A = \mathbf{env}_\Phi^{\mathbf{Bim}} A = \mathbf{env}_\Phi^\Omega A = \mathbf{env}_\Phi A, \quad \Omega \supseteq \mathbf{Bim}.$$

(b) in each algebra A there exist imprints of the classes **Mono** of all monomorphisms and **SMono** of all strong monomorphisms of the category \mathbf{Ste}^\otimes by means of arbitrary class of morphisms Φ , among which there is at least one coming to A ; in addition,

(i) if Φ differs morphisms on the inside in \mathbf{Ste}^\otimes , then the imprint of **Mono** is also an imprint of the class **Bim** of all bimorphisms:

$$\mathrm{imp}_\Phi^{\mathbf{Mono}} A = \mathrm{imp}_\Phi^{\mathbf{Bim}} A.$$

(ii) if Φ differs morphisms on the inside and is a left ideal in \mathbf{Ste}^\otimes , then the imprint of **Mono** is also an imprint of the class **Bim** of all bimorphisms, and of any other class Ω which contains **Bim** (for example, of the class **Mor** of all morphisms):

$$\mathrm{imp}_\Phi^{\mathbf{Mono}} A = \mathrm{imp}_\Phi^{\mathbf{Bim}} A = \mathrm{imp}_\Phi^\Omega A = \mathrm{imp}_\Phi A, \quad \Omega \supseteq \mathbf{Bim}.$$

- Let us say that a morphism of stereotype (or just topological) algebras $\varphi : A \rightarrow B$ is *dense*, if the image $\varphi(A)$ of A under the map φ is dense in B :

$$\overline{\varphi(A)} = B.$$

Certainly, each dense morphism is an epimorphism, so we will call such morphisms dense epimorphisms. The class of all dense epimorphisms in the category \mathbf{Ste}^\otimes (or in \mathbf{TopAlg}) will be denoted by **DEpi**. It is connected with the classes **Epi** of all epimorphisms and **SEpi** of all strong epimorphisms through the embeddings

$$\mathbf{SEpi} \subset \mathbf{DEpi} \subset \mathbf{Epi}. \quad (4.23)$$

Remark 4.3. Embeddings (4.23) are not equalities. An example of a dense epimorphism, which is not strong, is the embedding of the algebra $\mathcal{C}^\infty(M)$ of smooth functions into the algebra $\mathcal{C}(M)$ of continuous functions on a smooth manifold M (this embedding is a bimorphism of stereotype algebras, so if it were a strong epimorphism, it would automatically be an isomorphism, but this is not true). And an example of non-dense epimorphism is the standard embedding of the algebra $\mathcal{P}(\mathbb{C})$ of all polynomials on the complex plane \mathbb{C} into the algebra $\mathcal{P}(\mathbb{C}^\times)$ of Laurent polynomials on the punctured complex plane \mathbb{C}^\times (we endow $\mathcal{P}(\mathbb{C})$ and $\mathcal{P}(\mathbb{C}^\times)$ with the strongest locally convex topology).

For dense epimorphisms the first part of Theorem 4.16 is strengthened as follows:

Theorem 4.17. *In the category \mathbf{Ste}^\otimes of stereotype algebras each algebra A has an envelope in the class **DEpi** of all dense epimorphisms with respect to arbitrary class of morphisms Φ , among which there is at least one going from A ; in addition,*

(i) *if Φ differs morphisms on the outside in \mathbf{Ste}^\otimes , then the envelope in **DEpi** is also an envelope in the class **DBim** of all dense bimorphisms:*

$$\mathrm{env}_\Phi^{\mathbf{DEpi}} A = \mathrm{env}_\Phi^{\mathbf{DBim}} A,$$

(ii) *if Φ differs morphisms on the outside and is a right ideal in \mathbf{Ste}^\otimes , then the envelope in **DEpi** is also an envelope in the class **DBim** of all dense bimorphisms, and for each class Ω , which contains **DBim** (for example, in the class **Mor** of all morphisms):*

$$\mathrm{env}_\Phi^{\mathbf{DEpi}} A = \mathrm{env}_\Phi^{\mathbf{DBim}} A = \mathrm{env}_\Phi^\Omega A = \mathrm{env}_\Phi A, \quad \Omega \supseteq \mathbf{DBim}.$$

Proof. Consider the algebra A as a stereotype space. By Theorem 3.22 A has an envelope $\mathrm{env}_\Phi^{\mathbf{Epi}(\mathbf{Ste})} A$ in the class **Epi** of all epimorphisms in the category \mathbf{Ste} with respect to the class Φ . We have to verify that this object is an envelope $\mathrm{env}_\Phi^{\mathbf{DEpi}(\mathbf{Ste}^\otimes)} A$ in the class **DEpi** of all dense epimorphisms of the category \mathbf{Ste}^\otimes . For this we need to subsequently verify that the reasoning in Theorems 2.8, 2.9 and 2.11 work.

1. First we suppose that Φ consists of just one morphism φ . By Theorem 4.15 the elements of nodal decomposition $\varphi = \mathrm{im}_\infty \varphi \circ \mathrm{red}_\infty \varphi \circ \mathrm{coim}_\infty \varphi$ in the category \mathbf{Ste} of stereotype spaces are morphisms of stereotype algebras. As a corollary, the dense epimorphism $\varepsilon_{\max} = \mathrm{red}_\infty \varphi \circ \mathrm{coim}_\infty \varphi$ is also a morphism in \mathbf{Ste}^\otimes . It is an extension of A in $\mathbf{DEpi}(\mathbf{Ste}^\otimes)$ with respect to φ , due to diagram

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{red}_\infty \varphi \circ \mathrm{coim}_\infty \varphi} & \mathrm{Im}_\infty \varphi \\ & \searrow \varphi & \swarrow \mathrm{im}_\infty \varphi \\ & B & \end{array} \quad (4.24)$$

Let $\sigma : A \rightarrow N$ be another extension of A in $\mathbf{DEpi}(\mathbf{Ste}^\otimes)$ with respect to φ :

$$\begin{array}{ccc} A & \xrightarrow{\sigma} & N \\ \varphi \searrow & & \swarrow \exists! \nu \in \mathbf{Mor}(\mathbf{Ste}^\otimes) \\ & B & \end{array}$$

Then since σ is an epimorphism in \mathbf{Ste} , by Theorem 2.1 there exists a morphism $\nu' : N \rightarrow \mathbf{Im}_\infty \varphi$ in \mathbf{Ste} such that the following diagram is commutative:

$$\begin{array}{ccccc} A & \xrightarrow{\varphi} & B & & \\ \downarrow \text{coim}_\infty \varphi & \searrow \sigma & \nearrow \nu & & \downarrow \text{im}_\infty \varphi \\ & N & & & \\ & \searrow \nu' & & & \\ \text{Coim}_\infty \varphi & \xrightarrow{\text{red}_\infty \varphi} & \text{Im}_\infty \varphi & & \end{array}$$

The map ν' is a homomorphism of algebras, since ν and $\text{im}_\infty \varphi$ are homomorphisms, and $\text{im}_\infty \varphi$ is an injective map: on the one hand,

$$\text{im}_\infty \varphi(\nu'(1_N)) = \nu(1_N) = 1_B = \text{im}_\infty \varphi(1_{\text{Im}_\infty \varphi}) \Rightarrow \nu'(1_N) = 1_B,$$

and, on the other,

$$\begin{aligned} \text{im}_\infty \varphi(\nu'(x \cdot y)) &= \nu(x \cdot y) = \nu(x) \cdot \nu(y) = \text{im}_\infty \varphi(\nu'(x)) \cdot \text{im}_\infty \varphi(\nu'(y)) = \text{im}_\infty \varphi(\nu'(x) \cdot \nu'(y)) \Rightarrow \\ &\Rightarrow \nu'(x \cdot y) = \nu'(x) \cdot \nu'(y). \end{aligned}$$

The last diagram can be transformed as follows:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \searrow \sigma & & \nearrow \nu \\ & N & \\ \downarrow \nu' & & \downarrow \text{im}_\infty \varphi \\ & \text{Im}_\infty \varphi & \end{array}$$

$\varepsilon_{\max} = \text{red}_\infty \varphi \circ \text{coim}_\infty \varphi$

We see now that ν' is the morphism ν from diagram (1.6). Its uniqueness follows from the fact that σ is an epimorphism.

2. Then we consider the case when Φ is an arbitrary set of morphisms. We take the subset Φ_A in Φ consisting of morphisms going from A ,

$$\varphi \in \Phi_A \iff \varphi \in \Phi \ \& \ \text{Dom}(\varphi) = A.$$

The envelope with respect to Φ is the same as envelope with respect to Φ_A . After that we consider the product $\prod_{\varphi \in \Phi_A} \text{Dom}(\varphi)$ and the corresponding product of morphisms $\prod_{\varphi \in \Phi_A} \varphi : A \rightarrow \prod_{\varphi \in \Phi_A} \text{Dom}(\varphi)$. The envelope of A with respect to morphisms Φ_A is exactly the envelope of A with respect to one morphism $\prod_{\varphi \in \Phi_A} \varphi$, and we reduced the task to the previous case.

3. Finally, suppose Φ is an arbitrary class of morphisms (not necessarily a set). Then following the same reasoning as in Theorem 2.11 we replace the class Φ with a set M of morphisms which gives the same envelope. This is possible since, due to Theorem 4.15, in all the diagrams in the proof of Theorem 2.11 the arising morphisms are not only morphisms in the category \mathbf{Ste} , but also morphisms in the category \mathbf{Ste}^\otimes .

4. When the existence of envelope $\text{env}_\Phi^{\mathbf{DEpi}(\mathbf{Ste}^\otimes)} A$ is proved we apply the same reasoning as in Theorem 3.22. If Φ differs the morphisms on the outside, then by Theorem 1.2 the existence of the envelope $\text{env}_\Phi^{\mathbf{DEpi}} A$ automatically implies the existence of the envelope $\text{env}_\Phi^{\mathbf{DEpi} \cap \mathbf{Mono}} A = \text{env}_\Phi^{\mathbf{DBim}} A$ and their coincidence: $\text{env}_\Phi^{\mathbf{DEpi}} A = \text{env}_\Phi^{\mathbf{DBim}} A$. And if Φ differs morphisms and is a right ideal in \mathbf{Ste}^\otimes , then by Theorem 1.3 the existence of the envelope $\text{env}_\Phi^{\mathbf{DBim}} A$ (which is already proved) automatically implies that for each class $\Omega \supseteq \mathbf{DBim}$ the envelope $\text{env}_\Phi^\Omega A$ also exists and they coincide: $\text{env}_\Phi^{\mathbf{DBim}} A = \text{env}_\Phi^\Omega A$. \square

(c) The Arens-Michael envelope

We will finish the paper with two examples of envelopes in the categories of topological algebras. The first of them is the Arens-Michael envelope.

The Arens-Michael envelope in the category TopAlg of topological algebras. The term “Arens-Michael envelope” which we use here was introduced by A. Ya. Helemskii in [14]. The applications of this construction were considered in [27] and [3].

Recall that an absolutely convex closed neighborhood of zero U in a topological algebra A is said to be *submultiplicative*, if $U \cdot U \subseteq U$. To any such neighborhood of zero U in A one can assign a two-sided closed ideal $\text{Ker } U = \bigcap_{\varepsilon > 0} \varepsilon \cdot U$ in A and a quotient algebra $A/\text{Ker } U$ endowed with (not the quotient topology as one could expect, but) the topology of normed space with the unit ball $U + \text{Ker } U$. Then the completion $(A/\text{Ker } U)^\nabla$ is a Banach algebra, and we denote it by A/U and call it the *quotient algebra of A by the neighborhood of zero U* . The natural map from A into A/U

$$\begin{array}{ccccc} & & \rho_U & & \\ & \searrow & & \nearrow & \\ A & \xrightarrow{\tau_U} & A/\text{Ker } U & \xrightarrow{\nabla_{A/\text{Ker } U}} & (A/\text{Ker } U)^\nabla = A/U \end{array}$$

(where τ_U is a quotient map, and $\nabla_{A/\text{Ker } U}$ is the completion map) will be called the *Banach quotient map of A by the neighborhood of zero U* .

Denote by \mathcal{B} the class of all Banach quotient maps $\{\rho_U : A \rightarrow A/U\}$, where A runs over the class of topological algebras, and U the set of all submultiplicative neighborhoods of zero in A .

Proposition 4.1. *The class \mathcal{B} of Banach quotient maps is a net of epimorphisms in the category TopAlg of topological algebras, and the relation of pre-order¹² \rightarrow is equivalent to the embedding of the corresponding neighborhoods of zero up to a positive scalar multiplier:*

$$\rho_V \rightarrow \rho_U \iff \exists \varepsilon > 0 \quad \varepsilon \cdot V \subseteq U. \quad (4.25)$$

Proof. 1. Let us first verify (4.25). Suppose U and V are submultiplicative closed absolutely convex neighborhoods of zero in A , and $\varepsilon \cdot V \subseteq U$ for some $\varepsilon > 0$. Then $\text{Ker } V \subseteq \text{Ker } U$, and the formula

$$x + \text{Ker } V \mapsto x + \text{Ker } U$$

defines a linear continuous map $A/\text{Ker } V \rightarrow A/\text{Ker } U$ which can be extended by continuity to an operator

$$\pi_V^U : A/V = (A/\text{Ker } V)^\nabla \rightarrow (A/\text{Ker } U)^\nabla = A/U.$$

Obviously, the following diagram is commutative:

$$\begin{array}{ccc} & A & \\ \rho_V \swarrow & & \searrow \rho_U \\ A/V & \xrightarrow{\pi_V^U} & A/U \end{array}, \quad (4.26)$$

In particular, $\rho_V \rightarrow \rho_U$. On the contrary, if for some morphism $\iota : A/V \rightarrow A/U$ we have a commutative diagram

$$\begin{array}{ccc} & A & \\ \rho_V \swarrow & & \searrow \rho_U \\ A/V & \xrightarrow{\iota} & A/U \end{array}, \quad (4.27)$$

then we can put $\tilde{U} = \overline{\rho_U(U)}$ and $\tilde{V} = \overline{\rho_V(V)}$, and these will be ball centered in zeroes in A/U and A/V respectively, so the continuity of the operator $\iota : A/V \rightarrow A/U$ implies that

$$\varepsilon \cdot \tilde{V} \subseteq \iota^{-1}(\tilde{U})$$

¹²The pre-order \rightarrow on the class $\text{Epi}(X)$ of all epimorphisms going from a given object X of a category \mathbf{K} was defined on p.11.

for some $\varepsilon > 0$. And we have

$$\varepsilon \cdot V = (\rho_V)^{-1}(\varepsilon \cdot \tilde{V}) \subseteq (\rho_V)^{-1}(\iota^{-1}(\tilde{U})) = (\rho_U)^{-1}(\tilde{U}) = U.$$

2. Let us check now axiom (a) of the net of epimorphisms from the page 34. For each topological algebra A the set \mathcal{B}_A of its Banach quotient maps is non-empty, since always there exists at least one submultiplicative neighborhood of zero U in A , namely, $U = A$ (and the corresponding quotient map is zero, $\rho_U : A \rightarrow 0$). Besides this, if U and V are two submultiplicative closed absolutely convex neighborhoods of zero in A , then, clearly, its intersection $U \cap V$ is also a submultiplicative (and closed absolutely convex) neighborhood of zero in A . That is, the submultiplicative absolutely convex neighborhoods of zero form a system directed to the contraction in A . Together with the rule (4.25) this means that the system of epimorphisms $\{\rho_U : A \rightarrow A/U\}$ is directed to the left with respect to the pre-order \rightarrow .

3. Then we check axiom (b). For each topological algebra A the system of connecting morphisms $\text{Bind}(\mathcal{B}_A)$ has a projective limit, since the category \mathbf{TopAlg} is complete. This limit can be defined as a map $A \mapsto \varprojlim \text{Bind}(\mathcal{B}_A)$, since it is directly constructed as a set in the product of algebras A/U .

4. It remains to check axiom (c). Let $\alpha : A \rightarrow B$ be a morphism of topological algebras and $\rho_V : B \rightarrow B/V$ a Banach quotient map. The set $U = \alpha^{-1}(V)$ is a submultiplicative closed absolutely convex neighborhood of zero in A . The map

$$x + \text{Ker } U \mapsto \alpha(x) + \text{Ker } V,$$

is extended by continuity to a map $\alpha_U^V : A/U \rightarrow B/V$ such that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ \rho_U \downarrow & & \downarrow \rho_V \\ A/U & \xrightarrow{\alpha_U^V} & B/V \end{array}$$

□

- The net \mathcal{B} will be called a *net of Banach quotient maps*.
- For each algebra A diagram (4.26) means that the family of quotient maps $\rho_U : A \rightarrow A/U$ is a projective cone of the contravariant system $\text{Bind}(\mathcal{B}_A) = \{\pi_V^U\}$. The projective limit of this cone in the category \mathbf{TopAlg} of topological algebras is called the *Arens-Michael envelope* of the algebra A and is denoted by $\heartsuit_A : A \rightarrow A^\heartsuit$:

$$\heartsuit_A = \varprojlim \mathcal{B}_A. \quad (4.28)$$

(this limit exists since \mathbf{TopAlg} is complete). The range of this morphism

$$A^\heartsuit = \text{Ran}(\heartsuit_A) = \text{TopAlg-}\varprojlim_{\substack{U \text{ is a submultiplicative} \\ \text{neighborhood of zero in } A}} A/U. \quad (4.29)$$

is also called the *Arens-Michael envelope* of the algebra A .

For any two submultiplicative closed absolutely convex neighborhoods of zero U and V such that $\varepsilon \cdot V \subseteq U$ for some $\varepsilon > 0$ we obtain a diagram

$$\begin{array}{ccc} & A & \\ \rho_V \swarrow & \downarrow \heartsuit_A & \searrow \rho_U \\ & A^\heartsuit & \\ \pi_V \swarrow & & \searrow \pi_U \\ A/V & \xrightarrow{\pi_V^U} & A/U \end{array}$$

where $\heartsuit_A = \varprojlim_{U \rightarrow 0} \rho_U = \varprojlim \mathcal{B}_A$ is the natural morphism into the projective limit, or, what is the same, the local limit of the net \mathcal{B} at the element A . From Theorem 1.11 it follows

Theorem 4.18. *The Arens-Michael envelope is an envelope in TopAlg with respect to the system of Banach quotient maps \mathcal{B} , and to each morphism $\varphi : A \rightarrow B$ in TopAlg the formula*

$$\varphi^\heartsuit = \varprojlim_{\tau \in \mathcal{B}_B} \varprojlim_{\sigma \in \mathcal{B}_A} \varphi_\sigma^\tau \circ \sigma_B \quad (4.30)$$

assigns a morphism $\varphi^\heartsuit : A^\heartsuit \rightarrow B^\heartsuit$ such that the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\heartsuit_A} & A^\heartsuit \\ \downarrow \varphi & & \downarrow \varphi^\heartsuit \\ B & \xrightarrow{\heartsuit_B} & B^\heartsuit \end{array}, \quad (4.31)$$

and the map $(A, \varphi) \mapsto (A^\heartsuit, \varphi^\heartsuit)$ is a covariant functor from TopAlg into TopAlg .

Lemma 4.4. *In the category TopAlg of topological algebras the net \mathcal{B} of Banach quotient maps consists of dense epimorphisms and generates on the inside the class of all morphisms with values in Banach algebras.*

Proof. The class \mathcal{B} consists of dense epimorphisms, since the image $\rho_U(A)$ of any algebra A is always dense in its Banach quotient algebra $A/U = (A/\text{Ker } U)^\blacktriangledown$. Let us show that \mathcal{B} generates the class of morphisms with values in Banach algebras. It is important here to verify the second embedding in the chain (1.13). Let $\varphi : A \rightarrow B$ be a morphism into a Banach algebra B . If V is a unit ball in B , then the set $U = \varphi^{-1}(V)$ is a neighborhood of zero in A , and the condition $V \cdot V \subseteq V$ implies the condition $U \cdot U \subseteq U$:

$$x, y \in U \Rightarrow \varphi(x), \varphi(y) \in V \Rightarrow \varphi(x \cdot y) = \varphi(x) \cdot \varphi(y) \in V \Rightarrow x \cdot y \in U = \varphi^{-1}(V)$$

Consider the normed algebra $A/\text{Ker } U$ and the quotient map $\tau_U : A \rightarrow A/\text{Ker } U$. From the obvious equality $\text{Ker } \varphi = \text{Ker } U$ it follows that the morphism φ can be decomposed in the category Alg of algebras as follows:

$$\begin{array}{ccc} A & \xrightarrow{\tau_U} & A/\text{Ker } U \\ & \searrow \varphi & \downarrow \chi \\ & & B \end{array}$$

On the other hand, the equality $\chi^{-1}(V) = U + \text{Ker } \varphi = U + \text{Ker } U$ implies continuity of χ . So it will be continuously extended to the completion $(A/\text{Ker } U)^\blacktriangledown = A/U$ of the space $A/\text{Ker } U$:

$$\begin{array}{ccccc} & & \rho_U & & \\ & \curvearrowright & & \curvearrowright & \\ A & \xrightarrow{\tau_U} & A/\text{Ker } U & \xrightarrow{\blacktriangledown_{A/\text{Ker } U}} & A/U \\ & \searrow \varphi & \downarrow \chi & \swarrow \chi^\blacktriangledown & \\ & & B & & \end{array}$$

and since $A/\text{Ker } U$ is dense in its completion $(A/\text{Ker } U)^\blacktriangledown = A/U$, the map χ^\blacktriangledown must be multiplicative by Lemma 4.1. At the same time, obviously, χ^\blacktriangledown preserves the identity. Hence, χ^\blacktriangledown is a morphism in TopAlg . \square

Theorem 4.19. *In the category TopAlg of topological algebras the envelope with respect to the class BanAlg of Banach algebras exists for each algebra A in the following classes of morphisms:*

- in the class Epi of all epimorphisms,
- in the class DEpi of all dense epimorphisms,
- in the class SEpi of all strong epimorphisms.

In the first two cases the envelopes coincide with each other and with the Arens-Michael envelope:

$$\text{env}_{\text{BanAlg}}^{\text{Epi}} A = \text{env}_{\text{BanAlg}}^{\text{DEpi}} A = \heartsuit_A. \quad (4.32)$$

And in the third case the envelope coincide with the nodal coimage (in the category TopAlg) of the Arens-Michael envelope:

$$\text{env}_{\text{BanAlg}}^{\text{SEpi}} A = \text{coim}_\infty \heartsuit_A. \quad (4.33)$$

In all these cases the envelopes are covariant functors from TopAlg into TopAlg .

Proof. First, the topology in the projective limit (4.29) is the projective locally convex topology, hence the natural map $\heartsuit_A = \varprojlim_{U \rightarrow 0} \rho_U : A \rightarrow A^\heartsuit$ densely embeds the space A into the space A^\heartsuit . Thus, \heartsuit_A is a dense epimorphism. In addition, by Lemma 4.4 the net \mathcal{B} generates the class of morphisms with values in Banach algebras. Together this means that the conditions (i) and (ii) of Theorem 1.13 are fulfilled. Applying this theorem, we obtain that the local limit of the net \mathcal{B} , i.e. the Arens-Michael envelope, is an envelope in the class \mathbf{DEpi} of dense epimorphisms with respect to the class \mathbf{BanAlg} of Banach algebras:

$$\heartsuit_A = \varprojlim \mathcal{B}_A = \mathbf{env}_{\mathbf{BanAlg}}^{\mathbf{DEpi}} A.$$

After that we can replace the class \mathbf{DEpi} by the class \mathbf{Epi} in this reasoning, and then the same Theorem 1.13 will give the equality

$$\heartsuit_A = \varprojlim \mathcal{B}_A = \mathbf{env}_{\mathbf{BanAlg}}^{\mathbf{Epi}} A.$$

Finally, by Theorem 2.22 the nodal coimage of the local limit $\varprojlim \mathcal{B}_A$ must be an envelope in the class \mathbf{SEpi} of strong epimorphisms:

$$\mathbf{coim}_\infty \heartsuit_A = \mathbf{coim}_\infty \varprojlim \mathcal{B}_A = \mathbf{env}_{\mathbf{BanAlg}}^{\mathbf{SEpi}} A.$$

In all these cases the envelopes are functors by Theorems 1.13 and 2.22. \square

The following example shows that the net \mathcal{B} does not satisfy the conditions on page 40.

Example 4.3. The net \mathcal{B} of Banach quotient maps in the category \mathbf{TopAlg} is not relatively splitted.

Proof. Consider the algebra $\mathbb{C}[x]$ of polynomials of one variable with complex coefficients. Let $\alpha : \mathbb{C} \rightarrow \mathbb{C}[x]$ be the embedding of the field \mathbb{C} into $\mathbb{C}[x]$ by constants, $\tau : \mathbb{C}[x] \rightarrow \mathbb{C}$ a map that any polynomial f turns into its value in zero, $f(0)$, and $\delta : \mathbb{C}[x] \rightarrow \mathbb{C}$ a map that any polynomial f turns into its value in identity, $f(1)$. Then in the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\alpha} & \mathbb{C}[x] \\ \text{id}_{\mathbb{C}} \downarrow & \swarrow \delta & \downarrow \tau \\ \mathbb{C} & \xrightarrow{\text{id}_{\mathbb{C}}} & \mathbb{C} \end{array} \quad (4.34)$$

the upper triangle is commutative, since on the constants the value in the identity is the same as the value in the other points

$$\delta(\alpha(\lambda)) = \lambda, \quad \lambda \in \mathbb{C}.$$

And the lower triangle is not commutative, since, for instance, the values of the polynomial x in the points 0 and 1 are different:

$$\delta(x) = x(1) = 1 \neq 0 = x(0) = \tau(x).$$

It remains to note that τ is isomorphic to the Banach quotient map by the neighborhood of zero $V = \{f \in \mathbb{C}[x] : |f(0)| \leq 1\}$, and $\text{id}_{\mathbb{C}}$ to the Banach quotient map by the neighborhood of zero $U = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ (and moreover, all Banach quotient maps of \mathbb{C} are isomorphic to $\text{id}_{\mathbb{C}}$). As a corollary, if for τ a counterfort σ is chosen, then the corresponding diagram must be isomorphic to (4.34), and therefore it cannot split. \square

The Arens-Michael envelope in the category \mathbf{Ste}^\otimes of stereotype algebras. In the special case, when topological algebras are stereotype, the same construction as the one on page 111 defines a net of epimorphisms \mathcal{B} , which we also call a *net of Banach quotient maps*. But the local limit of this net $\mathbf{Ste}^\otimes\text{-}\varprojlim \mathcal{B}_A$ in the category \mathbf{Ste}^\otimes of stereotype algebras seems to not coincide with its local limit $\mathbf{TopAlg}\text{-}\varprojlim \mathcal{B}_A$ in the category \mathbf{TopAlg} of all topological algebras – the formal connection between them (actually, this is the common for all projective limits in \mathbf{TopAlg} and \mathbf{Ste}^\otimes , cf. [2, (4.15)]) is described by the formula

$$\mathbf{Ste}^\otimes\text{-}\varprojlim \mathbf{Bind}(\mathcal{B}_A) = \left(\mathbf{TopAlg}\text{-}\varprojlim \mathbf{Bind}(\mathcal{B}_A) \right)^\Delta.$$

We denote the projective limit of the cone of all quotient map $\rho_U : A \rightarrow A/U$ in the category \mathbf{Ste}^\otimes by the same symbol (as for the category \mathbf{TopAlg}), $\heartsuit_A : A \rightarrow A^\heartsuit$, and we call it the *stereotype Arens-Michael envelope* of the algebra A :

$$\heartsuit_A = \mathbf{Ste}^\otimes\text{-}\varprojlim \mathcal{B}_A = \left(\mathbf{TopAlg}\text{-}\varprojlim \mathcal{B}_A \right)^\Delta. \quad (4.35)$$

The range of this morphism, A^{\heartsuit} , is also called the *stereotype Arens-Michael envelope* of the algebra A :

$$A^{\heartsuit} = \text{Ran}(\heartsuit_A) = \left(\varprojlim_{\substack{U \text{ is a submultiplicative} \\ \text{neighborhood of zero in } A}} \text{TopAlg-lim } A/U \right)^{\Delta}. \quad (4.36)$$

The analog of Theorem 4.18 looks as follows:

Theorem 4.20. *The stereotype Arens-Michael envelope is an envelope in \mathbf{Ste}^{\otimes} with respect to the system of Banach quotient maps \mathcal{B} , and to each morphism $\varphi : A \rightarrow B$ in \mathbf{Ste}^{\otimes} the formula*

$$\varphi^{\heartsuit} = \varprojlim_{\tau \in \mathcal{B}_B} \varprojlim_{\sigma \in \mathcal{B}_A} \varphi_{\sigma}^{\tau} \circ \sigma_{\mathcal{B}} \quad (4.37)$$

assigns a morphism $\varphi^{\heartsuit} : A^{\heartsuit} \rightarrow B^{\heartsuit}$ such that the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\heartsuit_A} & A^{\heartsuit} \\ \downarrow \varphi & & \downarrow \varphi^{\heartsuit} \\ B & \xrightarrow{\heartsuit_B} & B^{\heartsuit} \end{array}, \quad (4.38)$$

and the map $(A, \varphi) \mapsto (A^{\heartsuit}, \varphi^{\heartsuit})$ is a covariant functor from \mathbf{Ste}^{\otimes} into \mathbf{Ste}^{\otimes} .

And the analog of Lemma 4.4 is the following:

Lemma 4.5. *In the category \mathbf{Ste}^{\otimes} of stereotype algebras the net \mathcal{B} of Banach quotient maps consists of dense epimorphisms and generates on the inside the class of morphisms with values in Banach algebras.*

Theorem 4.19 gets an analog with qualitative differences, since local local limits of the net of Banach quotient map apparently can be not epimorphisms (however, this question remains open):

Theorem 4.21. *In the category \mathbf{Ste}^{\otimes} of stereotype algebras the envelope with respect to the class \mathbf{BanAlg} of Banach algebras exists for each algebra A in the following classes of morphisms:*

- in the class \mathbf{Epi} of all epimorphisms, and then the envelope is connected to the stereotype Arens-Michael envelope by the formula

$$\text{env}_{\mathbf{BanAlg}}^{\mathbf{Epi}} A = \text{red}_{\infty} \heartsuit_A \circ \text{coim}_{\infty} \heartsuit_A, \quad (4.39)$$

where red_{∞} and coim_{∞} are elements of nodal decomposition in the category \mathbf{Ste}^{\otimes} of stereotype algebras,

- in the class \mathbf{DEpi} of all dense epimorphisms, and then the envelope is connected to the stereotype Arens-Michael envelope by the formula

$$\text{env}_{\mathbf{BanAlg}}^{\mathbf{DEpi}} A = \text{red}_{\infty} \heartsuit_A \circ \text{coim}_{\infty} \heartsuit_A, \quad (4.40)$$

where red_{∞} and coim_{∞} are elements of nodal decomposition in the category \mathbf{Ste}^{\otimes} of stereotype spaces (not algebras),

- in the class \mathbf{SEpi} of all strong epimorphisms, and then

$$\text{env}_{\mathbf{BanAlg}}^{\mathbf{SEpi}} A = \text{coim}_{\infty} \heartsuit_A, \quad (4.41)$$

where coim_{∞} is the element of nodal decomposition in the category \mathbf{Ste}^{\otimes} of stereotype algebras.

In all those cases the envelopes are covariant functor from \mathbf{Ste}^{\otimes} into \mathbf{Ste}^{\otimes} .

Proof. By Lemma 4.5 the net \mathcal{B} generates on the inside the class of morphisms with values in Banach algebras, so we can apply Theorem 2.13, and we obtain (4.39). The same reasoning give formula (4.40), but one need to notice here that the dense epimorphisms ε in \mathbf{Ste}^{\otimes} are exactly the morphisms which being considered as morphisms in the category \mathbf{Ste} of stereotype spaces, have degenerated nodal decomposition: $\text{im}_{\infty} \varepsilon \cong 1_{\text{Ran}(\varepsilon)}$ (the isomorphism in the category $\mathbf{Mono}(\text{Ran}(\varepsilon))$). Finally, for formula (4.41) we need to supplement Lemma 4.5 with Theorem 2.22. \square

Fourier transform on a commutative Stein group. Let G be a commutative compactly generated Stein group, $\mathcal{O}(G)$ the algebra of holomorphic functions on G , $\mathcal{O}^*(G)$ its dual space, considered as stereotype algebra with multiplication generated by the multiplication in G (see details in [3]), G^\bullet the dual group of complex characters on G , i.e. continuous homomorphisms $\chi : G \rightarrow \mathbb{C}^\times$ into the multiplicative group \mathbb{C}^\times of non-zero complex numbers (G^\bullet is endowed with the pointwise multiplication and with the topology of uniform convergence on compact sets in G), $\mathcal{F}_G : \mathcal{O}^*(G) \rightarrow \mathcal{O}(G^\bullet)$ the Fourier transform on G , i.e. the homomorphism of algebras acting by formula

$$\begin{array}{c} \text{value of the function } \mathcal{F}_G(\alpha) \in \mathcal{O}(G^\bullet) \\ \text{in the point } \chi \in G^\bullet \\ \downarrow \\ \overbrace{\mathcal{F}_G(\alpha)(\chi)} = \underbrace{\alpha(\chi)} \quad (\chi \in G^\bullet, \alpha \in \mathcal{O}^*(G)) \\ \uparrow \\ \text{action of the functional } \alpha \in \mathcal{O}^*(G) \\ \text{at the function } \chi \in G^\bullet \subseteq \mathcal{O}(G) \end{array}$$

Theorem 4.22. *For each compactly generated commutative Stein group G its Fourier transform $\mathcal{F}_G : \mathcal{O}^*(G) \rightarrow \mathcal{O}(G^\bullet)$ is the Arens-Michael envelope of the algebra $\mathcal{O}^*(G)$, and it coincides with the envelope with respect to the class of Banach algebras in the classes **Epi** of all epimorphisms and **DEpi** of all dense epimorphisms (both in the categories **TopAlg** and **Ste^{*}**):*

$$\mathcal{F}_G = \heartsuit_{\mathcal{O}^*(G)} = \text{env}_{\text{BanAlg}}^{\text{Epi}} \mathcal{O}^*(G) = \text{env}_{\text{BanAlg}}^{\text{DEpi}} \mathcal{O}^*(G). \quad (4.42)$$

Proof. In [3] it was proved that in **TopAlg**

$$\mathcal{F}_G = \heartsuit_{\mathcal{O}^*(G)}.$$

By formula (4.32) this implies the other equalities in the category **TopAlg**. In addition, $\mathcal{O}(G^\bullet)$ is a Fréchet algebra, hence from the fact that the morphism $\mathcal{F}_G : \mathcal{O}^*(G) \rightarrow \mathcal{O}(G^\bullet)$, as the local limit in **TopAlg**, is a dense epimorphism, it follows that it must be a dense epimorphism in **Ste^{*}** as well. That is $\text{im}_\infty \heartsuit_{\mathcal{O}^*(G)} \cong \text{id}_{\mathcal{O}(G^\bullet)}$, and therefore in formulas (4.39) and (4.40) the left sides can be replaced by the morphism $\heartsuit_{\mathcal{O}^*(G)}$. Thus, (4.42) is also true in **Ste^{*}**. \square

(d) C^* -envelopes

C^* -envelopes in the category **InvTopAlg of involutive topological algebras.**

- Let us say that a topological algebra A is *involutive*, if it has the operation of involution $x \mapsto \bar{x}$ (in the usual sense, cf. [14] or [23]), and this operation is continuous as a map from A into A . The involutive topological algebras form a category **InvTopAlg**, where morphisms are continuous involutive homomorphisms $\varphi : A \rightarrow B$ preserving identity:

$$\varphi(\lambda \cdot x + \mu \cdot y) = \lambda \cdot \varphi(x) + \mu \cdot \varphi(y), \quad \varphi(x \cdot y) = \varphi(x) \cdot \varphi(y), \quad \varphi(1) = 1, \quad \varphi(\bar{x}) = \overline{\varphi(x)}$$

Obvious examples of such algebras are usual C^* -algebras ([14], [23]). Another example is the algebra $C(M)$ of continuous functions on a paracompact locally compact topological space M with the topology of uniform convergence on compact sets, cf.[2].

- By C^* -seminorm on an involutive algebra A we mean any seminorm $p : A \rightarrow \mathbb{R}_+$ satisfying the following condition:

$$p(x \cdot \bar{x}) = p(x)^2, \quad x \in A. \quad (4.43)$$

By the Z. Sebestyen theorem [32], any such seminorm automatically preserves involution and is submultiplicative:

$$p(\bar{x}) = p(x), \quad p(x \cdot y) \leq p(x) \cdot p(y).$$

The identity (4.43) implies in particular the equality

$$p(1) = p(1 \cdot \bar{1}) = p(1)^2,$$

which mean that p must turn 1 either into 1, or into 0,

$$p(1) = 1 \quad \vee \quad p(1) = 0,$$

and the second one means that p vanishes, since in this case

$$p(x) = p(x \cdot 1) \leq p(x) \cdot p(1) = p(x) \cdot 0 = 0.$$

Further we will be interested in continuous C^* -seminorms on involutive topological algebras.

- Let us call a C^* -neighborhood in a topological algebra A any closed absolutely convex neighborhood of zero U , for which the Minkowski functional

$$p(x) = \inf\{\lambda > 0 : \lambda \cdot x \in U\}$$

is a C^* -seminorm on A . For any such neighborhood of zero U the quotient algebra A/U (defined on p.111) is a C^* -algebra, and we call it the C^* -quotient algebra of A , and the natural map $\rho_U : A \rightarrow A/U$ will be called a C^* -quotient map of A . The symbol C^* will denote the class of all C^* -quotient maps $\{\rho_U : A \rightarrow A/U\}$, where A runs over the class of involutive topological algebras, and U over the set of all C^* -neighborhoods of zero in A .

The following fact is an analog of Proposition 4.1.

Proposition 4.2. *The class C^* of all C^* -quotient maps is a net of epimorphisms in the category InvTopAlg of involutive topological algebras, and the semi-order \rightarrow in C^* is equivalent to the embedding of the neighborhoods of zero:*

$$\rho_V \rightarrow \rho_U \iff V \subseteq U. \quad (4.44)$$

Proof. By definition, the relation $\rho_V \rightarrow \rho_U$ means the existence of an involutive continuous homomorphism of C^* -algebras $\iota : A/V \rightarrow A/U$, such that diagram (4.27) is commutative. By the well-known property of C^* -algebras [23, Theorem 2.1.7], the homomorphism ι cannot increase the C^* -norm: $\|\iota(x)\| \leq \|x\|$. Being applied to C^* -seminorms p_U and p_V , which correspond to the neighborhoods U and V , this means inequality $p_U(x) \leq p_V(x)$, which in its turn is equivalent to embedding $V \subseteq U$. \square

- The net C^* will be called the *net of C^* -quotient maps*.
- For each involutive topological map A the family of C^* -quotient maps $\rho_U : A \rightarrow A/U$ is a projective cone of the covariant system $\text{Bind}(C_A^*)$. The projective limit of this cone in the category InvTopAlg of involutive topological algebras is called the C^* -envelope of the algebra A and is denoted¹³ $\diamond_A : A \rightarrow A^\diamond$:

$$\diamond_A = \varprojlim C_A^*. \quad (4.45)$$

(this limit always exists, since the category InvTopAlg is complete). The range of this morphism

$$A^\diamond = \text{Ran}(\diamond_A) = \text{InvTopAlg-} \varprojlim_{\substack{U \text{ is a} \\ C^*\text{-neighborhood of zero in } A}} A/U. \quad (4.46)$$

is also called a C^* -envelope of the algebra A .

Theorem 1.11 implies

Theorem 4.23. *The C^* -envelope is an envelope in InvTopAlg with respect to the system C^* of C^* -quotient maps, and to each morphism $\varphi : A \rightarrow B$ in InvTopAlg the formula*

$$\varphi^\diamond = \varprojlim_{\tau \in C_B^*} \varprojlim_{\sigma \in C_A^*} \varphi_\sigma^\tau \circ \sigma_{C^*} \quad (4.47)$$

assigns a morphism $\varphi^\diamond : A^\diamond \rightarrow B^\diamond$ such that the following diagram is commutative,

$$\begin{array}{ccc} A & \xrightarrow{\diamond_A} & A^\diamond \\ \downarrow \varphi & & \downarrow \varphi^\diamond \\ B & \xrightarrow{\diamond_B} & B^\diamond \end{array}, \quad (4.48)$$

and the map $(A, \varphi) \mapsto (A^\diamond, \varphi^\diamond)$ is a covariant functor from InvTopAlg into InvTopAlg .

Lemma 4.6. *In the category InvTopAlg of involutive topological algebras the net C^* of all C^* -quotient maps consists of dense epimorphisms and generates on the inside the class of morphisms with values in C^* -algebras.*

¹³We use here the notation of Yu. N. Kuznetsova from [21].

Proof. Suppose $\varphi : A \rightarrow B$ is a morphism into a C^* -algebra B . Take the unit ball V in B , and consider its preimage $U = \varphi^{-1}(V)$. This is a neighborhood of zero in A , and its Minkowski functional p coincides with the composition of φ and the norm on B :

$$p(x) = \inf\{\lambda > 0 : \lambda \cdot x \in \varphi^{-1}(V)\} = \inf\{\lambda > 0 : \lambda \cdot \varphi(x) \in V\} = \|\varphi(x)\|.$$

This means that p is a C^* -seminorm on A :

$$p(x \cdot \bar{x}) = \|\varphi(x \cdot \bar{x})\| = \left\| \varphi(x) \cdot \overline{\varphi(x)} \right\| = \|\varphi(x)\|^2 = p(x)^2.$$

In other words, U is a C^* -neighborhood of zero in A . Then we repeat the proof of Lemma 4.4. \square

The following proposition is proved by analogy with Theorem 4.19:

Theorem 4.24. *In the category InvTopAlg of involutive topological algebras the envelope with respect to the class \mathcal{C}^* of all C^* -algebras exists for each algebra A in the following classes of morphisms:*

- in the class Epi of all epimorphisms,
- in the class DEpi of all dense epimorphisms,
- in the class SEpi of all strong epimorphisms.

In the first two cases the envelopes coincide with each other and with the C^* -envelope:

$$\text{env}_{\mathcal{C}^*}^{\text{Epi}} A = \text{env}_{\mathcal{C}^*}^{\text{DEpi}} A = \diamond_A. \quad (4.49)$$

And in the third case the envelope coincide with the nodal coimage of the C^* -envelope:

$$\text{env}_{\mathcal{C}^*}^{\text{SEpi}} A = \text{coim}_{\infty} \diamond_A. \quad (4.50)$$

In all these cases the envelopes are covariant functors from InvTopAlg into InvTopAlg .

The C^* -envelopes in the category InvSte^{\otimes} of involutive stereotype algebras. As in the case of the Arens-Michael envelopes, in the full subcategory InvSte^{\otimes} of involutive stereotype algebras of the category InvTopAlg the properties of the C^* -envelopes change since the local limit $\text{InvSte}^{\otimes}\text{-}\varprojlim \mathcal{C}_A^*$ of the net \mathcal{C}^* in the category InvSte^{\otimes} may not coincide with the local limit $\text{InvTopAlg}\text{-}\varprojlim \mathcal{C}_A^*$ in InvTopAlg . The connection between them is the following:

$$\text{InvSte}^{\otimes}\text{-}\varprojlim \text{Bind}(\mathcal{C}_A^*) = \left(\text{InvTopAlg}\text{-}\varprojlim \text{Bind}(\mathcal{C}_A^*) \right)^{\Delta}.$$

To avoid new notations, we will use the same symbol $\diamond_A : A \rightarrow A^{\diamond}$ for the projective limit of the projective cone of C^* -quotient maps $\rho_U : A \rightarrow A/U$ in the category InvSte^{\otimes} (as we did it for the category InvTopAlg). We call it the *stereotype C^* -envelope* of the algebra A :

$$\diamond_A = \text{Ste}^{\otimes}\text{-}\varprojlim \mathcal{C}_A^* = \left(\text{TopAlg}\text{-}\varprojlim \mathcal{C}_A^* \right)^{\Delta}. \quad (4.51)$$

The range of this morphism A^{\diamond} will also be called the *stereotype C^* -envelope* of the algebra A :

$$A^{\diamond} = \text{Ran}(\diamond_A) = \left(\text{TopAlg}\text{-}\varprojlim_{\substack{U \text{ is a} \\ C^*\text{-neighborhood of zero in } A}} A/U \right)^{\Delta}. \quad (4.52)$$

The analogs of Theorems 4.23, 4.24 and of Lemma 4.6 look here as follows:

Theorem 4.25. *The stereotype C^* -envelope is an envelope in InvSte^{\otimes} with respect to the system \mathcal{C}^* of C^* -quotient maps, and to each morphism $\varphi : A \rightarrow B$ in InvSte^{\otimes} the formula*

$$\varphi^{\diamond} = \varprojlim_{\tau \in \mathcal{C}_B^*} \varinjlim_{\sigma \in \mathcal{C}_A^*} \varphi_{\sigma}^{\tau} \circ \sigma_{\mathcal{C}^*} \quad (4.53)$$

assigns a morphism $\varphi^{\diamond} : A^{\diamond} \rightarrow B^{\diamond}$ such that the following diagram is commutative,

$$\begin{array}{ccc} A & \xrightarrow{\diamond_A} & A^{\diamond} \\ \downarrow \varphi & & \downarrow \varphi^{\diamond} \\ B & \xrightarrow{\diamond_B} & B^{\diamond} \end{array}, \quad (4.54)$$

and the map $(A, \varphi) \mapsto (A^{\diamond}, \varphi^{\diamond})$ is a covariant functor from InvSte^{\otimes} into InvSte^{\otimes} .

Lemma 4.7. *In the category InvSte^{\otimes} of involutive stereotype algebras the net C^* of all C^* -quotient maps generates on he inside the class of morphisms with values in C^* -algebras.*

Theorem 4.26. *In the category InvSte^{\otimes} of involutive stereotype algebras the envelope with respect to the class C^* of all C^* -algebras exists for each algebra A in the following classes of morphisms:*

- in the class Epi of all epimorphisms, and then the envelope is connected to the stereotype C^* -envelope through the formula

$$\text{env}_{C^*}^{\text{Epi}} A = \text{red}_{\infty} \diamond_A \circ \text{coim}_{\infty} \diamond_A, \quad (4.55)$$

where red_{∞} and coim_{∞} are elements of the nodal decomposition in the category InvSte^{\otimes} of involutive stereotype algebras,

- in the class DEpi of all dense epimorphisms, and the the envelope is connected with the stereotype C^* -envelope through the formula

$$\text{env}_{C^*}^{\text{DEpi}} A = \text{red}_{\infty} \diamond_A \circ \text{coim}_{\infty} \diamond_A, \quad (4.56)$$

where red_{∞} and coim_{∞} are elements of the nodal decomposition in the category Ste of stereotype spaces (not algebras),

- in the class SEpi of all strong epimorphisms, and then

$$\text{env}_{C^*}^{\text{SEpi}} A = \text{coim}_{\infty} \diamond_A, \quad (4.57)$$

where coim_{∞} is the element of nodal decomposition in the category InvSte^{\otimes} of involutive stereotype algebras.

In all these cases the envelopes are covariant functors from InvSte^{\otimes} into InvSte^{\otimes} .

The Gelfand transform as a C^* -envelope of a commutative algebra.

- By *involutive spectrum* $\text{Spec}(A)$ of an involutive topological (respectively, stereotype) algebra A over \mathbb{C} we mean the set of its involutive characters, i.e. homomorphisms $\chi : A \rightarrow \mathbb{C}$ (also continuous, involutive and preserving identity). This set is endowed with the topology of uniform convergence on the totally bounded sets in A .
- By *Gelfand transform* of an involutive stereotype algebra A we mean the natural map $\mathcal{G}_A : A \rightarrow C(M)$ of A into the algebra $C(M)$ of functions on the involutive spectrum $M = \text{Spec}(A)$, continuous on each compact set $K \subseteq M$:

$$\mathcal{G}_A(x)(t) = t(x), \quad t \in M = \text{Spec}(A), \quad x \in A. \quad (4.58)$$

We endow algebra $C(M)$ with the topology which is a pseudosaturation¹⁴ of the topology of uniform convergence on compact sets in M – this turns $C(M)$ into a stereotype algebra. In the special case, when M is a paracompact locally compact space, the topology of uniform convergence on compact sets in M is already a pseudosaturated (and complete) topology on $C(M)$, so $C(M)$ becomes a stereotype algebra already at this step [2, Sec.8.1] (and the operation of pseudosaturation do not change this topology anymore).

- For each compact set $K \subseteq M$ let us consider the restriction map

$$\pi_K : C(M) \rightarrow C(K), \quad y \mapsto y|_K,$$

and let $\mathcal{G}_K = \pi_K \circ \mathcal{G}$ be the composition

$$\begin{array}{ccc} A & \xrightarrow{\gamma} & C(M) \\ & \searrow \mathcal{G}_K & \swarrow \pi_K \\ & C(K) & \end{array} \quad (4.59)$$

If K and L are two compact sets in M , and $K \subseteq L \subseteq M$, then by symbol π_K^L we denote the restriction map

$$\pi_K^L : C(L) \rightarrow C(K), \quad y \mapsto y|_K.$$

¹⁴The operation of pseudosaturation was defined in [2, Sec. 1.4].

Obviously, the algebra $C(M)$ with the system of projections $\rho_K : C(M) \rightarrow C(K)$, $K \subseteq M$, is a projective limit of the system of binding morphisms $\pi_K^L : C(L) \rightarrow C(K)$, $K \subseteq L \subseteq M$ (in the category \mathbf{InvSte}^\otimes):

$$C(M) = \mathbf{InvSte}^\otimes\text{-}\varprojlim_{K \subseteq M} C(K).$$

Proposition 4.3. *For any involutive stereotype algebra A its Gelfand transform $\mathcal{G}_A : A \rightarrow C(M)$ is a morphism of stereotype algebras. In the special case when the spectrum $M = \mathbf{Spec}(A)$ of A is a paracompact locally compact space, the morphism $\mathcal{G}_A : A \rightarrow C(M)$ is a dense epimorphism.*

Proof. In the first part of this proposition only the continuity of the map \mathcal{G}_A is not obvious. Take a base neighborhood of zero U in $C(M)$, i.e. $U = \{f \in C(M) : \sup_{t \in T} |f(t)| \leq \varepsilon\}$ for some compact set $T \subseteq M$ and some $\varepsilon > 0$. Its preimage under the map $\mathcal{G}_A : A \rightarrow C(M)$ is the set $\{x \in A : \sup_{t \in T} |t(x)| \leq \varepsilon\} = \varepsilon \cdot {}^\circ T$, i.e. the homothety of the polar ${}^\circ T$ of the compact set T . Since A is stereotype, ${}^\circ T$ is a neighborhood of zero in it. This proves that the map $\mathcal{G}_A : A \rightarrow C(M)$ is continuous if the space $C(M)$ is endowed with the topology of uniform convergence on compact sets in M . Since the space A , being stereotype, is pseudosaturated, this means that under the pseudosaturation of the topology in $C(M)$ the map $\mathcal{G}_A : A \rightarrow C(M)$ remains continuous (this follows, for example, from [2, Theorem 1.16]).

Suppose further that $M = \mathbf{Spec}(A)$ is a paracompact locally compact space. For each compact set $K \subseteq M$ the image $\mathcal{G}_K(A)$ of the algebra A in $C(K)$ under the map \mathcal{G}_K is an involutive subalgebra in $C(K)$, and it contains the identity (and hence, all constant functions) and differs the points $t \in K$. So by the Stone-Weierstrass theorem, $\mathcal{G}_K(A)$ is dense in $C(K)$. This is true for each map $\mathcal{G}_K = \pi_K \circ \gamma$, where K is a compact set in M . Since the topology in $C(M)$ is the projective topology with respect to the maps π_K , we have that the image $\mathcal{G}_A(A)$ of A in $C(M)$ is dense in $C(M)$. \square

Theorem 4.27. *For each commutative involutive stereotype algebra A the system of morphisms $\mathcal{G}_K : A \rightarrow C(K)$ consists of dense epimorphisms and is isomorphic in the category $\mathbf{Epi}(A)$ to the system $\rho_U : A \rightarrow A/U$ of all C^* -quotient maps of A ,*

$$\{\mathcal{G}_K : A \rightarrow C(K), K \subseteq \mathbf{Spec}(A)\} \cong \mathcal{C}_A^*. \quad (4.60)$$

Under this isomorphism

- the system of restrictions $\pi_K^L : C(L) \rightarrow C(K)$, $K \subseteq L \subseteq M$ turns into the system $\mathbf{Bind}(\mathcal{C}_A^*)$ of binding morphisms of the net \mathcal{C}^* on the algebra A :

$$\{\pi_K^L : C(L) \rightarrow C(K), K \subseteq L \subseteq \mathbf{Spec}(A)\} \cong \mathbf{Bind}(\mathcal{C}_A^*). \quad (4.61)$$

- the Gelfand transform $\mathcal{G}_A : A \rightarrow C(M)$ is a local limit of the net \mathcal{C}^* on the algebra A (and hence, it coincides with the stereotype C^* -envelope of the algebra A):

$$\mathcal{G}_A = \varprojlim \mathcal{C}_A^* = \diamond_A \quad (4.62)$$

Proof. On each compact set $K \subseteq M$ the algebra of functions of the form $\mathcal{G}_A(x)$, where $x \in A$, differs the points, contains constant functions, and is invariant with respect to involution, so it is dense in $C(K)$ by the Stone-Weierstrass theorem. This implies, that the algebra $C(M)$, which contains A , is also dense in $C(K)$, so both morphisms $\mathcal{G}_K : A \rightarrow C(K)$ and $\pi_K : A \rightarrow C(K)$ are dense epimorphisms (in the category \mathbf{InvSte}^\otimes).

The range A/U of each C^* -quotient map $\rho_U : A \rightarrow A/U$ must be a commutative C^* -algebra, hence it is isomorphic to the algebra $C(T_U)$ of continuous functions on its spectrum T_U . Under the dual map $\rho_U^* : \mathbf{Spec}(A) \leftarrow \mathbf{Spec}(A/U)$ this spectrum T_U is homeomorphically turned into a compact set $K_U = \rho_U^*(T_U)$ in the space $M = \mathbf{Spec}(A)$, and we get the following diagram

$$\begin{array}{ccc} & A & \\ \rho_U \swarrow & & \searrow \mathcal{G}_{K_U} \\ A/U & \xrightarrow{\mathcal{G}_U} & C(K_U) \end{array}$$

where \mathcal{G}_U is the Gelfand transform of the algebra A/U in composition with the map, dual to the homeomorphism $T_U \cong K_U$.

On the contrary, for each compact set $K \subseteq M$ the set

$$U_K = \{a \in A : \sup_{t \in K} |t(a)| \leq 1\}$$

is a C^* -neighborhood of zero in A . The corresponding quotient algebra A/U_K will be commutative, hence it is isomorphic to the algebra $C(T_K)$ of continuous functions on its spectrum T_K , which is in addition homeomorphic to K . If we denote by \mathcal{G}_K the composition of the Gelfand transform of A with the dual map to the homeomorphism $T_K \cong K$, we obtain a commutative diagram

$$\begin{array}{ccc} & A & \\ \rho U_K \swarrow & & \searrow \mathcal{G}_K \\ A/U_K & \xrightarrow{\quad} & C(K) \end{array}.$$

Together this proves (4.60), and (4.61) and (4.62) become its obvious corollaries. \square

Lemma 4.8. *If the spectrum $M = \text{Spec}(A)$ of a stereotype algebra A is a k -space, then for each extension $\sigma : A \rightarrow C$ in the class Mor of all morphisms (in InvSte^\otimes) with respect to the class of C^* -algebras the dual map of spectra*

$$\sigma^* : \text{Spec}(C) \rightarrow \text{Spec}(A) = M \quad \Bigg| \quad \sigma(s) = s \circ \sigma, \quad s \in \text{Spec}(C)$$

is a homeomorphism of topological spaces.

Proof. First, the map σ^* must be an injection, since if some characters $s \neq s' \in \text{Spec}(C)$ have the same image under the action of σ^* , i.e.

$$s \circ \sigma = \sigma^*(s) = \sigma^*(s') = s' \circ \sigma,$$

then this can be understood in such a way that the character $s \circ \sigma = s' \circ \sigma : A \rightarrow \mathbb{C}$ has two different continuations on C :

$$\begin{array}{ccc} A & \xrightarrow{\sigma} & C \\ & \searrow & \swarrow s \\ & \mathbb{C} & \nwarrow s' \end{array}$$

$s \circ \sigma = s' \circ \sigma$

This is impossible, since σ is an extension, in particular, with respect to the C^* -algebra \mathbb{C} .

On the other hand, the map σ^* is a covering, i.e. for each compact set K in M there is a compact set T in $\text{Spec}(C)$ such that $\sigma^*(T) \supseteq K$. Indeed, if K is a compact set in $M = \text{Spec}(A)$, then, since $\sigma : A \rightarrow C$ is an extension with respect to the class of C^* -algebras, the natural homomorphism $\mathcal{G}_K : A \rightarrow C(K)$ into the C^* -algebra $C(K)$ have a continuation to C , i.e. a diagram arises:

$$\begin{array}{ccc} A & \xrightarrow{\sigma} & C \\ & \searrow \mathcal{G}_K & \swarrow \tau_K \\ & C(K) & \end{array}$$

If we now put $T = \tau_K^*(K)$, then

$$\sigma^*(T) = \sigma^*(\tau_K^*(K)) = \mathcal{G}_K^*(K) = K.$$

In addition, from the fact that σ^* is a covering, it follows that it is surjective. We obtain that $\sigma^* : \text{Spec}(C) \rightarrow \text{Spec}(A)$ is a continuous bijective covering. Since $\text{Spec}(A)$ is a k -space, the map σ^* is open, and thus, a homeomorphism. \square

The following result supplements the results of Yu. N. Kuznetsova's paper [21]:

Theorem 4.28. *If A is a commutative involutive stereotype algebra with the paracompact locally compact involutive spectrum $M = \text{Spec}(A)$, then its C^* -envelope (in the category InvSte^\otimes) in the classes of all morphisms, all epimorphisms and all dense epimorphisms is the algebra $C(M)$ of continuous functions on M :*

$$A^\diamond = \text{Env}_{C^*} A = \text{Env}_{C^*}^{\text{Epi}} A = \text{Env}_{C^*}^{\text{DEpi}} A = C(M)$$

Proof. The equality $A^\diamond = C(M)$ is already stated in Theorem 4.27. The other two ones with Epi and DEpi are proved quite simply (after the preparatory work which has already been done). For the class Epi the reasoning are as follows. By Theorem 4.27, the morphism \mathcal{G}_A is a local limit of the net \mathcal{C}^* on the object A , and by Proposition 4.3 it is an epimorphism:

$$\varprojlim \mathcal{C}_A^* = \mathcal{G}_A \in \text{Epi}.$$

On the other hand, by Lemma 4.7 the net \mathcal{C}^* generates on the inside the class of morphisms with values in C^* -algebras. Together this means that one can apply Theorem 1.13, and we will obtain that \mathcal{G}_A is an envelope in the class **Epi** of all epimorphisms with respect to the class of C^* -algebras:

$$\text{env}_{\mathcal{C}^*}^{\text{Epi}} A = \varprojlim \mathcal{C}_A^* = \mathcal{G}_A.$$

Similarly, if we replace **Epi** by **DEpi**, we obtain the equality

$$\text{env}_{\mathcal{C}^*}^{\text{DEpi}} A = \varprojlim \mathcal{C}_A^* = \mathcal{G}_A.$$

Thus, it remains only to prove the equality where **Epi** is replaced by the class **Mor** of all morphisms in **InvSte**[®]:

$$\text{Env}_{\mathcal{C}^*} A = C(M).$$

Let us show first that $\mathcal{G}_A : A \rightarrow C(M)$ is an extension of the algebra A with respect to the class of C^* -algebras. Suppose that $\varphi : A \rightarrow B$ is a morphism of A into a C^* -algebra B . For constructing the dotted arrow φ' in diagram (1.5),

$$\begin{array}{ccc} A & \xrightarrow{\mathcal{G}_A} & C(M) \\ & \searrow \varphi & \swarrow \varphi' \\ & B & \end{array}$$

we can think that B is commutative and that $\varphi(A)$ is dense in B (since otherwise we can replace B by the closure $\overline{\varphi(A)}$ in B , and this will be a commutative subalgebra in B). Then from the commutativity of B we deduce that B is of the form $C(K)$, and from the density of $\varphi(A)$ in B – that the compact set K is injectively embedded into $M = \text{Spec}(A)$. So our diagram can be represented as follows:

$$\begin{array}{ccc} A & \xrightarrow{\mathcal{G}_A} & C(M) \\ & \searrow \mathcal{G}_K & \swarrow \varphi' \\ & C(K) & \end{array}$$

where K is a compact subset in M , and \mathcal{G}_K is defined in (4.59). Clearly, one can take as φ' the restriction map π_K from M to K , which we considered before:

$$\begin{array}{ccc} A & \xrightarrow{\mathcal{G}_A} & C(M) \\ & \searrow \mathcal{G}_K & \swarrow \pi_K \\ & C(K) & \end{array}$$

In this way the dotted arrow will be unique, since by Proposition 4.3 \mathcal{G}_A is an epimorphism.

Let us now check that $\mathcal{G}_A : A \rightarrow C(M)$ is a maximal extension, i.e. for any other extension $\sigma : A \rightarrow C$ there is a unique morphism $v : C \rightarrow C(M)$ such that the following diagram is commutative:

$$\begin{array}{ccc} & A & \\ \sigma \swarrow & & \searrow \mathcal{G}_A \\ C & \dashrightarrow_v & C(M) \end{array} \quad (4.63)$$

By Lemma 4.8, the dual map of spectra $\sigma^* : \text{Spec}(C) \rightarrow \text{Spec}(A) = M$ is a homeomorphism. Hence, a map is defined

$$v : C \rightarrow C(M) \quad \Bigg| \quad v(y)(t) = \underbrace{(\sigma^*)^{-1}(t)}_{\cap \text{Spec}(C)}(y), \quad y \in C, \quad t \in M.$$

It is verified elementary that this is a morphism of involutive topological algebras. In addition, diagram (4.63) is commutative:

$$v(\sigma(x))(t) = (\sigma^*)^{-1}(t)(\sigma(x)) = \sigma^*((\sigma^*)^{-1}(t))(x) = t(x) = \mathcal{G}_A(x)(t), \quad x \in A, \quad t \in M$$

i.e. $v \circ \sigma = \mathcal{G}_A$.

It remains to verify that the dotted arrow in (4.63) is unique. Suppose that v' is another dotted arrow with the same properties:

$$v \circ \sigma = \mathcal{G}_A = v' \circ \sigma. \quad (4.64)$$

If v and v' are different, they do not coincide on some vector $y \in C$:

$$v(y) \neq v'(y).$$

Here in both sides there are functions on M , so the inequality means that they do not coincide in some point $t \in M$:

$$v(y)(t) \neq v'(y)(t).$$

Put

$$s(z) = v(z)(t), \quad s'(z) = v'(z)(t), \quad z \in C,$$

then we see that two different characters on C give a same character in composition with σ :

$$s(\sigma(x)) = v(\sigma(x))(t) = (4.64) = v'(\sigma(x))(t) = s'(\sigma(x)), \quad x \in A.$$

By Lemma 4.8 this is impossible, so our initial supposition that $v \neq v'$ is also not true. \square

Fourier transform on a commutative locally compact group. Let G be a commutative locally compact group, $\mathcal{C}(G)$ the algebra of continuous functions on G , $\mathcal{C}^*(G)$ the dual stereotype space, considered as a stereotype algebra with the multiplication generated by the multiplication on G (see details e.g. in [2]), G^\bullet the dual group of characters on G , i.e. continuous homomorphisms $\chi : G \rightarrow \mathbb{T}$ into the circle \mathbb{T} (G^\bullet is endowed with the pointwise algebraic operations and the topology of uniform convergence on compact sets in G), $\mathcal{F}_G : \mathcal{C}^*(G) \rightarrow \mathcal{C}(G^\bullet)$ the Fourier transform on G , i.e. the homomorphism of algebras, acting by formula

$$\begin{array}{ccc} \text{value of the function } \mathcal{F}_G(\alpha) \in \mathcal{C}(G^\bullet) \\ \text{in the point } \chi \in G^\bullet & & \\ \downarrow & & \\ \overbrace{\mathcal{F}_G(\alpha)(\chi)} & = & \underbrace{\alpha(\chi)} \quad (\chi \in G^\bullet, \quad \alpha \in \mathcal{C}^*(G)) \\ & \uparrow & \\ & \text{action of the functional } \alpha \in \mathcal{C}^*(G) \\ & \text{at the function } \chi \in G^\bullet \subseteq \mathcal{C}(G) & \end{array}$$

The following observation belongs to Yu. N. Kuznetsova [21]:

Theorem 4.29. *For each commutative locally compact group G its Fourier transform $\mathcal{F}_G : \mathcal{C}^*(G) \rightarrow \mathcal{C}(G^\bullet)$ is a C^* -envelope of the algebra $\mathcal{C}^*(G)$, and coincides with the envelope with respect to the class of C^* -algebras in the classes **Mor** of all morphisms, **Epi** of all epimorphisms and **DEpi** of all dense epimorphisms of involutive stereotype algebras:*

$$\mathcal{F}_G = \diamond_{\mathcal{C}^*(G)} = \text{env}_{C^*} \mathcal{C}^*(G) = \text{env}_{C^*}^{\text{Epi}} \mathcal{C}^*(G) = \text{env}_{C^*}^{\text{DEpi}} \mathcal{C}^*(G).$$

Proof. The spectrum of the algebra $\mathcal{C}^*(G)$ is homeomorphic to G^\bullet , so everything follows from Theorem 4.28. \square

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